integral $\int_a^b f(t)K(t,s)\,dt$ is a function of the single variable s, so denoting the integral by $F(s)$ we can write

$$F(s) = \int_a^b K(t,s) f(t)\,dt. \tag{1}$$

The function $F(s)$ in (1) is called an **integral transform** of $f(t)$, the function $K(t,s)$ is the **kernel** of the transform, and s is the **transform variable**. The limits a and b may be finite or infinite, and when at least one limit is infinite the integral in (1) becomes an improper integral.

When it exists, the **Laplace transform** $F(s)$ of a real function $f(t)$ with domain of definition $0 \le t < \infty$ is defined as the integral transform (1) with the kernel $K(t,s) = e^{-st}$, the interval of integration $0 \le t < \infty$, and s a complex variable such that Re $s < c$ for some nonnegative constant c, so that

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt. \tag{2}$$

Throughout the present chapter the transform variable s will be considered to be a real variable, and c will be chosen such that the integral in (2) converges. However, when the general problem of recovering a function $f(t)$ from its Laplace transform $F(s)$ is considered in Chapter 16, it will be seen that s must be allowed to be a complex variable. The advantage of restricting s to the real variable case in this chapter is that the recovery of many useful and frequently occurring functions $f(t)$ from their Laplace transforms $F(s)$ can be accomplished in a very simple manner without the use of complex variable methods.

The reason for interest in integral transforms in general, and the Laplace transform in particular, will become clear when the solution of initial value problems for differential equations is considered. It will then be seen that the Laplace transform replaces integrations with respect to t by simple algebraic operations involving $F(s)$, so provided $f(t)$ can be recovered from $F(s)$ in a simple manner, the solution of an initial value problem can be found by means of straightforward algebraic operations.

Clearly the kernel e^{-st} will only decrease as t increases if $s > 0$, and the Laplace transform of $f(t)$ will only be defined for functions $f(t)$ that decrease sufficiently rapidly as $t \to \infty$ for the integral in (2) to exist. In general, if the function to be transformed is denoted by a lowercase letter such as f, then its Laplace transform will be denoted by the corresponding uppercase letter F, as in (2). It is convenient to denote the Laplace transform operation by the symbol $\mathcal{L}$, so that symbolically $F(s) = \mathcal{L}\{f(t)\}$.

The Laplace transform

formal definition of the Laplace transform

Let $f(t)$ be defined for $0 \le t < \infty$. Then, when the improper integral exists, the Laplace transform $F(s)$ of $f(t)$, written symbolically $F(s) = \mathcal{L}\{f(t)\}$, is defined as

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

EXAMPLE 7.1 Find $\mathcal{L}\{e^{at}\}$ where a is real.

Solution From (2) we have

$$\begin{aligned}\mathcal{L}\{e^{at}\} &= \int_0^\infty e^{-st}e^{at}\,dt \\ &= \left[\frac{-e^{-(s-a)t}}{s-a}\right]_0^{t\to\infty} \\ &= \lim_{t\to\infty}\left[\frac{-e^{-(s-a)t}}{s-a}\right] + \frac{1}{s-a} \\ &= \frac{1}{s-a},\end{aligned}$$

provided $s > a$, for only then will the limit in the first term vanish. This has shown that $\mathcal{L}\{e^{at}\} = F(s) = 1/(s-a)$ for $s > a$, where it is necessary to include the inequality $s > a$ to ensure the convergence of the integral. ■

PIERRE SIMON LAPLACE (1749–1827)
A French mathematician of remarkable ability who made contributions to analysis, differential equations, probability, and celestial mechanics. He used mathematics as a tool with which to investigate physical phenomena, and made fundamental contributions to hydrodynamics, the propagation of sound, surface tension in liquids, and many other topics. His many contributions had a wide-ranging effect on the development of mathematics.

Laplace transform pair and inverse transform

The two functions $f(t)$ and $F(s)$ are called a **Laplace transform pair**, and for all ordinary functions, given $F(s)$ the corresponding function $f(t)$ is determined uniquely, just as $f(t)$ determines $F(s)$ uniquely. This relationship is expressed symbolically by using the symbol $\mathcal{L}^{-1}$ to denote the operation of finding a function $f(t)$ with a given Laplace transform $F(s)$. This process is called finding the **inverse Laplace transform** of $F(s)$. In terms of the foregoing example, we have $\mathcal{L}\{e^{at}\} = 1/(s-a)$ and $\mathcal{L}^{-1}\{1/(s-a)\} = e^{at}$. This is a particular case of the general result that, by definition, the inverse Laplace transform acting on the Laplace transform of the function returns the original function, so we can write

$$\mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}\} = f(t).$$

how to be sure a Laplace transform exists

A sufficient condition for the existence of the Laplace transform of a function $f(t)$ is that the absolute value of $f(t)$ can be bounded for all $t \geq 0$ by

$$|f(t)| \leq Me^{kt}, \tag{3}$$

for some constants M and k. This means that if numbers M and k can be found such that

$$|e^{-st}f(t)| \leq Me^{(k-s)t},$$

then

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt \leq M\int_0^\infty e^{(k-s)t}dt = M/(s-k).$$

TABLE 7.1 Laplace Transform Pairs

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$	Condition on s
1. 1	$1/s$	$s > 0$
2. t	$1/s^2$	$s > 0$
3. $t^n\ (n = 1, 2, \dots)$	$n!/s^{n+1}$	$s > 0$
4. $t^a\ (a > -1)$	$\Gamma(a+1)/s^{a+1}$	$s > a$
5. e^{at}	$1/(s-a)$	$s > a$
6. $t^n e^{at}\ (n = 1, 2, \dots)$	$n!/(s-a)^{n+1}$	$s > a$
7. $H(t-a)$	e^{-as}/s	$s \geq a$
8. $\delta(t-a)$	e^{-as}	$s > 0, a > 0$
9. $\sin at$	$a/(s^2+a^2)$	$s > 0$
10. $\cos at$	$s/(s^2+a^2)$	$s > 0$
11. $t \sin at$	$2as/(s^2+a^2)^2$	$s > 0$
12. $t \cos at$	$(s^2-a^2)/(s^2+a^2)^2$	$s > 0$
13. $e^{at} \sin bt$	$b/[(s-a)^2+b^2]$	$s > a$
14. $e^{at} \cos bt$	$(s-a)/[(s-a)^2+b^2]$	$s > a$
15. $\dfrac{1}{2a^3} \sin at - \dfrac{1}{2a^2} t \cos at$	$1/(s^2+a^2)^2$	$s > 0$
16. $\dfrac{1}{2a} \sin at + \dfrac{1}{2} t \cos at$	$s^2/(s^2+a^2)^2$	$s > 0$
17. $1 - \cos at$	$a^2/[s(s^2+a^2)]$	$s > 0$
18. $at - \sin at$	$a^3/[s^2(s^2+a^2)]$	$s > 0$
19. $\sinh at$	$a/(s^2-a^2)$	$s > \lvert a \rvert$
20. $\cosh at$	$s/(s^2-a^2)$	$s > \lvert a \rvert$
21. $\dfrac{1}{2a^3} \sinh at + \dfrac{1}{2a^2} t \cosh at$	$1/(s^2-a^2)^2$	$s > \lvert a \rvert$
22. $\dfrac{1}{2a} t \sinh at$	$s/(s^2-a^2)^2$	$s > \lvert a \rvert$
23. $\dfrac{1}{2a} \sinh at + \dfrac{1}{2} t \cosh at$	$s^2/(s^2-a^2)^2$	$s > \lvert a \rvert$
24. $\sinh at - \sin at$	$2a^3/(s^4-a^4)$	$s > \lvert a \rvert$
25. $\cosh at - \cos at$	$2a^2 s/(s^4-a^4)$	$s > \lvert a \rvert$

The integral on the right will be convergent provided $s > k > 0$, so when this is true the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ will exist. It should be clearly understood that (3) is only a *sufficient* condition for the existence of a Laplace transform, and *not* a necessary one, because Laplace transforms can be found for functions that do not satisfy condition (3). For example, the function $f(t) = t^{-1/4}$ does not satisfy condition (3), but its Laplace transform exists and is a special case of entry 4 in Table 7.1.

The preceding inequality implies that when $\mathcal{L}\{f(t)\}$ exists, $F(s)$ must be such that $\lim_{s\to\infty} F(s) = 0$. In addition, the condition $\mathcal{L}\{f(t)\} \leq M/(s-k)$ implies that $F(s)$ cannot be the Laplace transform of on ordinary function $f(t)$ unless $F(s) \to 0$ as $s \to \infty$. For example, $F(s) = (s^2-1)/(s^2+1)$ is not a Laplace transform of an ordinary function. Exceptions to this condition are functions like the *delta function*, which is defined in Section 7.2, though there the delta function will be seen to involve integration, and so it is not a *function* in the usual sense.

The Laplace transform is a linear operation, and the consequence of this important and useful property is expressed in the following theorem.

THEOREM 7.1

fundamental linearity property

Linearity of the Laplace transformation Let the functions $f_1(t), f_2(t), \ldots, f_n(t)$ have Laplace transforms, and let $c_1, c_2, \ldots, c_n$ be any set of arbitrary constants. Then

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\} + \cdots + c_n\mathcal{L}\{f_n(t)\}.$$

Proof The proof is simple and follows directly from the fact that integration is a linear operation, so the integral of a sum of functions is the sum of their integrals. Thus,

$$\begin{aligned}&\int_0^\infty e^{-st}\{c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t)\}dt\\ &\quad = c_1\int_0^\infty f_1(t)e^{-st}dt + c_2\int_0^\infty f_2(t)e^{-st}dt + \cdots + c_n\int_0^\infty f_n(t)e^{-st}dt\\ &\quad = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\} + \cdots + c_n\mathcal{L}\{f_n(t)\}.\end{aligned}$$ ■

This theorem has many applications and its use is essential when working with the Laplace transform.

EXAMPLE 7.2

Find the Laplace transform of $f(t) = c_1e^{at} + c_2e^{-at}$, and use the result to find $\mathcal{L}\{\sinh at\}$ and $\mathcal{L}\{\cosh at\}$.

some examples

Solution Applying Theorem 7.1 and the result $\mathcal{L}\{e^{at}\} = 1/(s-a)$ from Example 7.1, we find that

$$\mathcal{L}\{c_1e^{at} + c_2e^{-at}\} = c_1\mathcal{L}\{e^{at}\} + c_2\mathcal{L}\{e^{-at}\} = c_1/(s-a) + c_2/(s+a).$$

As $\sinh at = (e^{at} - e^{-at})/2$ and $\cosh at = (e^{at} + e^{-at})/2$, $\mathcal{L}\{\sinh at\}$ is obtained from the preceding result by setting $c_1 = 1/2$ and $c_2 = -1/2$, and $\mathcal{L}\{\cosh at\}$ is obtained by setting $c_1 = c_2 = 1/2$, when we obtain

$$\mathcal{L}\{\sinh at\} = a/(s^2 - a^2) \quad \text{and} \quad \mathcal{L}\{\cosh at\} = s/(s^2 - a^2),$$

for $s > |a| \geq 0$. Notice that because s must be be positive, but in sinh at and cosh at the number a may be either positive or negative, the relationship between s and a necessary to ensure that the convergence of the integrals must be $s > |a| \geq 0$, and not $s > a > 0$. ■

The process of finding an inverse Laplace transformation involves reversing the foregoing argument and seeking a function $f(t)$ that has the required Laplace transform $F(s)$. Where possible, this is accomplished by simplifying the algebraic structure of $F(s)$ to the point at which it can be recognized as the sum of the Laplace transforms of known functions of t.

EXAMPLE 7.3

Find the inverse Laplace transform of

$$F(s) = \frac{4s+10}{s^2+6s+8}.$$

Solution Expanding the Laplace transform in terms of partial fractions gives

$$\frac{4s+10}{s^2+6s+8} = \frac{1}{s+2} + \frac{3}{s+4},$$

so

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{4s+10}{s^2+6s+8}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}.$$

Using the result of Example 7.1 we find that

$$f(t) = \mathcal{L}^{-1}\left\{\frac{4s+10}{s^2+6s+8}\right\} = e^{-2t} + 3e^{-4t}.$$

■

EXAMPLE 7.4 Find (a) $\mathcal{L}\{1\}$ and (b) $\mathcal{L}\{t\}$.

Solution

(a) By definition,

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}dt = \frac{1}{s}, \quad \text{for } s > 0.$$

(b) By definition,

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}t\,dt = \left(-\frac{t}{s}e^{-st} - \frac{e^{-st}}{s^2}\right)\Bigg|_{t=0}^{\infty} = \frac{1}{s^2}, \quad \text{for } s > 0.$$

■

EXAMPLE 7.5 Find $\mathcal{L}\{\sin at\}$.

Solution By definition,

$$\begin{aligned}\mathcal{L}\{\sin at\} &= \int_0^\infty e^{-st}\sin at\,dt = \lim_{k\to\infty}\int_0^k e^{-st}\sin at\,dt \\ &= \lim_{k\to\infty}\left(\frac{-e^{-sk}(a\cos ak + s\sin ak)}{s^2+a^2}\right) + \frac{a}{s^2+a^2} \\ &= \frac{a}{s^2+a^2} \quad \text{for } s > 0,\end{aligned}$$

where the condition $s > 0$ is required to ensure that the limit is finite as $k \to 0$. This has shown that

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2} \quad \text{for } s > 0.$$

■

In the next example we find $\mathcal{L}\{t^n\}$, and in the process introduce an integral that will be useful later in Chapter 8 when finding series solutions of linear second order variable coefficient differential equations.

EXAMPLE 7.6 Find $\mathcal{L}\{t^n\}$ for $n = 1, 2, \ldots$.

Solution By definition

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st}t^n\mathrm{d}t.$$

To evaluate this integral we will make use of integration by parts to establish a recursion (recurrence) relation from which the result for arbitrary positive integral n can be found.

Accordingly, we define $I(n, s)$ as

$$I(n, s) = \int_0^\infty e^{-st} t^n dt = \lim_{k\to\infty} \int_0^k \frac{-t^n}{s} \frac{d}{dt}(e^{-st}) dt$$

and use integration by parts to express this as

$$= \lim_{k\to\infty} \left[\frac{-t^n e^{-st}}{s}\right]_{t=0}^{k} + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st} dt$$

$$= \left(\frac{n}{s}\right) I(n-1, s), \quad \text{for } s > 0.$$

This has established the *recursion relation*

$$I(n, s) = (n/s) I(n-1,\ s),$$

satisfied by the integral $I(n, s)$.

As $I(0, s) = \int_0^\infty e^{-st} dt = 1/s$, by setting $n = 1$ in the recursion relation we find that

$$I(1, s) = (1/s) I(0, s) = 1/s^2, \quad \text{for } s > 0.$$

Similarly, setting $n = 2$ in the recursion relation shows that

$$I(2, s) = (2/s) I(1, s) = 2 \cdot 1/s^3 = 2!/s^3, \quad \text{for } s > 0,$$

and an inductive argument shows that

$$I(n, s) = n!/s^{n+1}.$$

In terms of the Laplace transform notation, we have shown that

$$\mathcal{L}\{t^n\} = n!/s^{n+1} \quad \text{for } n = 0,\ 1,\ 2, \ldots, \quad \text{for } s > 0.$$

■

Notice that setting $s = 1$ in the general result of Example 7.3 enables $n!$ to be expressed as the integral

$$n! = \int_0^\infty e^{-t} t^n dt, \quad \text{for } n = 0,\ 1,\ 2, \ldots.$$

first encounter with the Gamma function

This provides a way of representing factorial n in terms of an integral, and it is our first encounter with a special case of the **Gamma function** that will be required later. The gamma function, denoted by $\Gamma(x)$ for $x > 0$, is defined by the integral

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt. \tag{4}$$

In terms of the earlier notation, when the restriction that n is an integer is removed, and n is replaced by a positive real variable x, we can write

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x dt = I(x, 1),$$

but

$$I(x, 1) = xI(x-1, 1) = x\Gamma(x) \quad \text{for } x > 0,$$

so combining results shows that the gamma function satisfies the fundamental relation

$$\Gamma(x+1) = x\Gamma(x) \quad \text{for } x > 0. \tag{5}$$

It is easily seen from this that

$$\Gamma(n+1) = n! \quad \text{for } n = 0, 1, 2, \ldots,$$

so as $\Gamma(x)$ is defined for all positive x the gamma function provides a generalization of the factorial function $n!$ for positive non-integer values of n. It will be seen later that the gamma function, which belongs to the general class of functions called **higher transcendental functions**, occurs frequently throughout mathematics.

Discontinuous Functions

Because the Laplace transform is defined in terms of an integral, it is possible to find Laplace transforms of discontinuous functions. Suppose, for example, that a function $g(t)$ is discontinuous at $t = a$, as in Fig. 7.1. Then, provided it converges, the integral defining the Laplace transform of $g(t)$ is given by

$$\mathcal{L}\{g(t)\} = \lim_{\varepsilon \to 0} \int_0^{a-\varepsilon} e^{-st} g(t)\,dt + \lim_{\delta \to 0} \int_{a+\delta}^{\infty} e^{-st} g(t)\,dt, \tag{6}$$

where ε and δ are both positive. For simplicity, the upper limit in the first integral is usually denoted by a_- and the lower limit in the second integral by a_+. These are, respectively, the limits of integration to the left and right of $t = a$.

An important discontinuous function that finds numerous applications in connection with the Laplace transform, and elsewhere, is the **unit step function** $f(t) = H(t-a)$ with $a \geq 0$, known also as the **Heaviside step function**. The unit step function is defined as

Heaviside step function

$$H(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases} \qquad (a \geq 0). \tag{7}$$

A related function that is also of considerable importance is the **unit pulse function**,

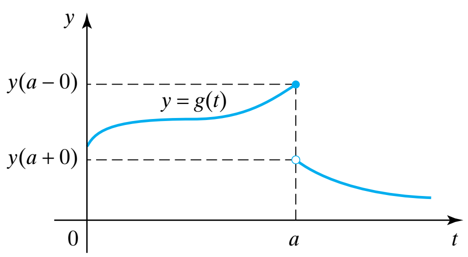


FIGURE 7.1 A discontinuous function $g(t)$.

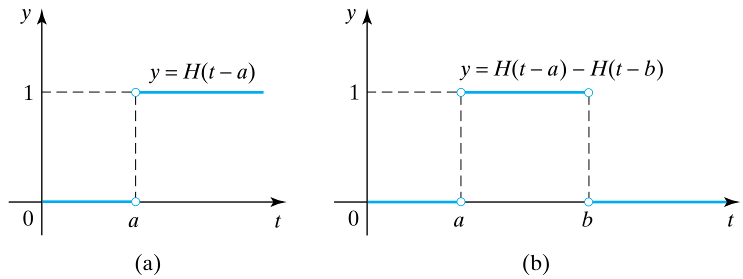


FIGURE 7.2 (a) The unit step function $y = H(t - a)$. (b) The unit pulse function $y = p(t) = H(t - a) - H(t - b)$.

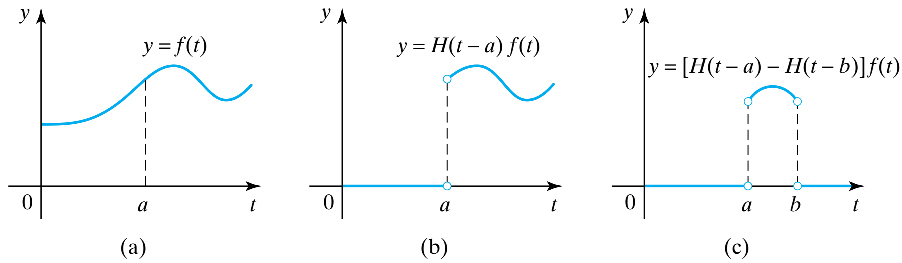


FIGURE 7.3 The effect on $f(t)$ of multiplication by $H(t - a)$ and $H(t - a) - H(t - b)$.

defined as

$$p(t) = H(t - a) - H(t - b), \quad \text{with } b > a \geq 0. \tag{8}$$

The function $p(t)$ operates like a "switch," because it switches on at $t = a$ and off at $t = b$. Graphs of these two functions are shown in Fig. 7.2.

switching functions on and off with the Heaviside step function

If a function $f(t)$ is multiplied by a unit step function, the function $f(t)$ can be considered to be "switched on" at time $t = a$, in the sense that the product $H(t - a)f(t)$ is zero for $t < a$ and $f(t)$ for $t > a$. Similarly, multiplication of $f(t)$ by a unit pulse function "switches on" the function $f(t)$ at time $t = a$ and "switches it off" at time $t = b$. This property is illustrated in Fig. 7.3, where Fig. 7.3(a) shows the original function $f(t)$, Fig. 7.3(b) shows the product $H(t - a)f(t)$, and Fig. 7.3(c) the product $\{H(t - a) - H(t - b)\}f(t)$.

In the next example we make use of result (6) to find the Laplace transforms of the unit step function and the unit pulse function.

EXAMPLE 7.7

Find (a) $\mathcal{L}\{H(t - a)\}$ and (b) $\mathcal{L}\{H(t - a) - H(t - b)\}$.

Solution

(a) By definition

$$\mathcal{L}\{H(t - a)\} = \int_a^\infty e^{-st}\,dt$$

$$= \left(-\frac{e^{-st}}{s}\right)\Bigg|_{t=a}^{\infty} = \frac{e^{-as}}{s} \quad \text{for } s > a \geq 0.$$

(b) Using result (a) we have

$$\begin{aligned}\mathcal{L}\{H(t-a)-H(t-b)\} &= \int_a^b e^{-st}\,dt \\ &= \int_a^\infty e^{-st}\,dt - \int_b^\infty e^{-st}\,dt \\ &= \frac{e^{-as}-e^{-bs}}{s} \quad \text{for } s > b > a \geq 0.\end{aligned}$$

■

EXAMPLE 7.8 Find (a) $\mathcal{L}\{t^3 - 4t + 5 + 3\sin 2t\}$ and (b) $\mathcal{L}^{-1}\{(s^4+5s^2+2)/[s^3(s^2+1)]$.

Solution

(a) Using Theorem 7.1 together with the Laplace transform pairs found in the previous examples, we have

$$\begin{aligned}\mathcal{L}\{t^3 - 4t + 5 + 2\sin 3t\} &= \mathcal{L}\{t^3\} - 4\mathcal{L}\{t\} + \mathcal{L}\{5\} + 3\mathcal{L}\{\sin 2t\} \\ &= 6/s^4 - 4/s^2 + 5/s + 6/(s^2+4) \\ &= (5s^5 + 2s^4 + 20s^3 - 10s^2 + 24)/[s^4(s^2+4)].\end{aligned}$$

(b) Simplifying the transform by means of partial fractions gives

$$\frac{s^4+5s^2+2}{s^3(s^2+1)} = \frac{2}{s^3} + \frac{3}{s} - 2\frac{s}{s^2+1}.$$

Taking the inverse Laplace transform of each term on the right and using the linearity property of the Laplace transform, we find that

$$\mathcal{L}^{-1}\left(\frac{s^4+5s^2+2}{s^3(s^2+1)}\right) = \mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}.$$

Finally, using the transform pairs established in the previous examples, we have

$$\mathcal{L}^{-1}\left\{\frac{s^4+5s^2+2}{s^3(s^2+1)}\right\} = t^2 + 3 - 2\cos t.$$

■

To make further progress with the Laplace transform it is necessary to have available a table of Laplace transform pairs for the most commonly occurring functions. Theorems to be developed later will enable such a table to be extended in a straightforward manner, so that transforms and inverse Laplace transforms of more complicated functions can be found.

Table 7.1 provides a list of the most useful Laplace transform pairs involving elementary functions. All of these entries can be established either by means of routine integration, or by the combination of simpler results, with the sole exception of the *delta function* $\delta(t-a)$ in entry 8. The derivation of this result is to be found in Section 7.2 after the delta function has been defined.

The example that now follows illustrates how entry 15 can be found from entries 9 through 12.

EXAMPLE 7.9 Find $\mathcal{L}^{-1}\{1/(s^2+a^2)^2\}$ by combining related entries in Table 7.1.

Solution Our objective will be to use the linearity property of the Laplace transform to express $1/(s^2+a^2)^2$ as a linear combination of terms that we hope will be found listed in the column $F(s)$ of Table 7.1. If this is possible, the inverse Laplace transform can then be found by adding the inverse transform of each expression in partial fraction representation of $F(s)$. A routine calculation shows that $F(s)$ can be written as

$$\frac{1}{(s^2+a^2)^2} = \frac{1}{2a^3}\left(\frac{a}{s^2+a^2}\right) - \frac{1}{2a^2}\left(\frac{s^2-a^2}{(s^2+a^2)^2}\right),$$

so from using entries 9 and 12 in Table 7.1 we have

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \frac{1}{2a^3}\sin at - \frac{1}{2a^2}t\cos at,$$

and this is entry 15 in the table. ■

Summary

The Laplace transform of a function $f(t)$ has been defined. A condition has been given that ensures the existence of the transform, and the concept of a Laplace transform pair has been introduced. The transform has been shown to have the fundamental property of linearity, and some simple transform pairs have been found directly from the definition. The Heaviside unit step function $H(t-a)$, which jumps from zero for $0 \le t < a$ to unity for $t > a$, has been introduced and used. The section closed with a table of useful Laplace transform pairs.

EXERCISES 7.1

In Exercises 1 through 4 use the definition of the Laplace transform to obtain the stated result.

1. Show that $\mathcal{L}\{t^2\} = 2/s^3$ for $s > 0$.
2. Show that $\mathcal{L}\{te^{at}\} = 1/(s-a)^2$ for $s > a$.
3. Find $\mathcal{L}\{e^{iat}\}$, and by equating the real and imaginary parts show that $\mathcal{L}\{\sin at\} = a/(s^2+a^2)$ and $\mathcal{L}\{\cos at\} = s/(s^2+a^2)$ for $s > 0$.
4. Show that $\mathcal{L}\{\sinh at\} = a/(s^2-a^2)$ for $s > |a|$.

In Exercises 5 through 20 use Table 7.1 of Laplace transform pairs to find $\mathcal{L}\{f(t)\}$.

5. $f(t) = te^{2t}$.
6. $f(t) = 2\sin 3t - \cos 3t$.
7. $f(t) = t - t^2 + t^3$.
8. $f(t) = e^{3t}(\sin t - \cos t)$.
9. $f(t) = e^{-2t}(\cos 2t - \sin 2t)$.
10. $f(t) = t(\sin 2t - \cos 2t)$.
11. $f(t) = t\cosh 3t - \sinh 3t$.
12. $f(t) = \sinh t - t\cos t$.
13. $f(t) = e^{-t}\cos 2t - t$.
14. $f(t) = 2t^2 - 3t + 4\cos 3t$.
15. $f(t) = H(t-\pi/2)e^t\sin t$.
16. $f(t) = H(t-3\pi/2)(\sin t - 3\cos t)$.
17. $f(t) = [H(t-\pi/2) - H(t-\pi)]t$.
18. $f(t) = [1 - H(t-\pi/2)]t$.
19. $f(t) = H(t-\pi/2)e^{-t}\cos t$.
20. $f(t) = [1 - H(t-\pi/2)]e^{3t}$.

In Exercises 21 through 30 use Table 7.1 of Laplace transform pairs to find $\mathcal{L}^{-1}\{F(s)\}$.

21. $F(s) = (s^2-1)/[s(s^2+4)]$.
22. $F(s) = (s^2+3s+1)/[s(s^2-4)]$.
23. $F(s) = (3s+5)/[s(s^2+9)]$.
24. $F(s) = (s^2-4)/[(s^2+1)(s^2-1)]$.
25. $F(s) = (s^3-1)/[(s+2)^2(s^2-9)]$.
26. $F(s) = (s^2+s+1)/[(s^2+4)(s^2-9)]$.
27. $F(s) = s^2/[(s-1)^2(s+1)]$.
28. $F(s) = s/(s-1)^3$.
29. $F(s) = (s^2+4)/[(s^2-9)(s-1)]$.
30. $F(s) = (s^2+1)/[(s+1)(s+2)(s+3)$.

In Exercises 31 through 36 find the Laplace transform of the function $f(t)$ shown in graphical form.

31.

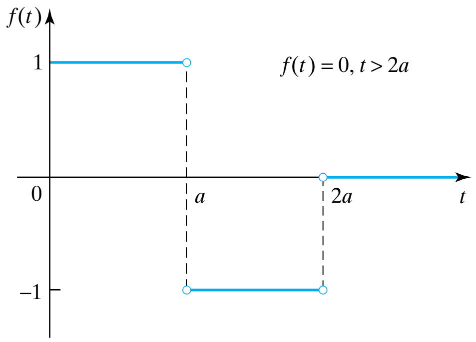


FIGURE 7.4

32.

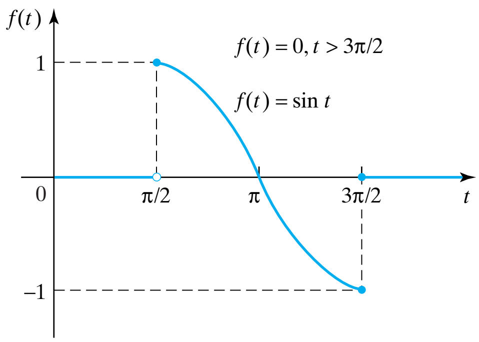


FIGURE 7.5

33.

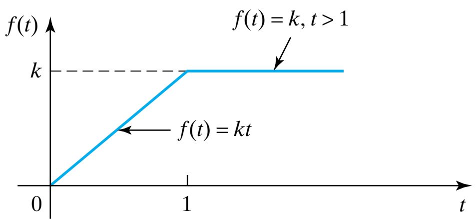


FIGURE 7.6

34.

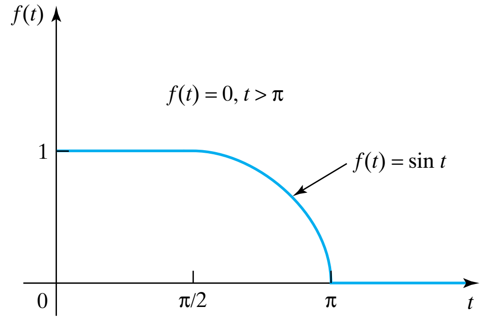


FIGURE 7.7

35.

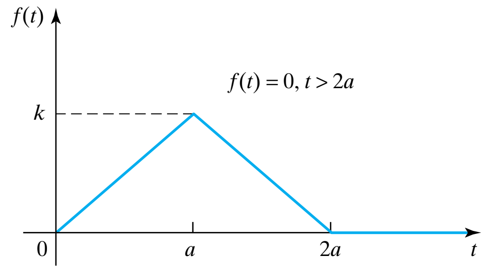


FIGURE 7.8

36.

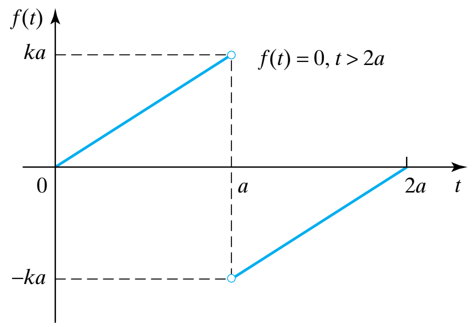


FIGURE 7.9

7.2 Operational Properties of the Laplace Transform

In the previous section the Laplace transform of a basic list of commonly occurring functions $f(t)$ was recorded as the list of Laplace transform pairs in Table 7.1. To use the Laplace transform to solve initial value problems for linear differential equations and systems it is necessary to establish a number of fundamental properties of the transform known as its **operational properties**. This name is given to properties of the transform itself that relate to the way it *operates* on any function $f(t)$ that is transformed, rather than to the effect these properties of the transform have on specific functions $f(t)$.

This means that operational properties are general properties of the Laplace transform that are not specific to any particular function $f(t)$ or to its transform

$F(s)$. An important example of an operational property has already been encountered in Theorem 7.1, where the linearity property of the transformation was established.

Some operational properties, such as the scaling and shift theorems that will be proved later, save effort when finding the Laplace transform of a function or inverting a transform, whereas others such as the transform of a derivative are essential when applying the Laplace transform to solve initial value problems for differential equations.

The way derivatives transform is used to find how the homogeneous part of a linear differential equation is transformed, and we will see later that it also shows how the initial conditions for the differential equation enter into the transformed equation. Table 7.1 of Laplace transform pairs is needed when transforming the nonhomogeneous term in the differential equation.

THEOREM 7.2

transforming derivatives

Transform of a derivative Let $f(t)$ be continuous on $0 \le t < \infty$, and let $f'(t)$ be piecewise continuous on every finite interval contained in $t \ge 0$. Then if $\mathcal{L}\{f(t)\} = F(s)$,

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0).$$

Proof Using integration by parts, and assuming that f satisfies the sufficiency condition for the existence of a Laplace transform, we have

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= \int_0^\infty e^{-st} f'(t)dt = \lim_{k\to\infty}\int_0^k e^{-st} f'(t)dt \\ &= \lim_{k\to\infty}[e^{-st} f(t)]_0^k - \lim_{k\to\infty}\int_0^k -se^{-st} f(t)dt \\ &= \lim_{k\to\infty}[e^{-sk} f(k) - f(0)] + sF(s) \\ &= sF(s) - f(0),\end{aligned}$$

where $\lim_{k\to\infty} e^{-sk} f(k) = 0$ because of condition (3). ∎

THEOREM 7.3

Transform of a higher derivative Let $f(t)$ be continuous on $0 \le t < \infty$, and let $f'(t), f''(t), \ldots, f^{(n-1)}(t)$ be piecewise continuous on every finite interval contained in $t \ge 0$. Then if $\mathcal{L}\{f(t)\} = F(s)$,

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Proof The proof uses repeated integration by parts, but otherwise is analogous to the one used in Theorem 7.2, so the details are left as an exercise. ∎

The two most frequently used results are those of Theorem 7.2 and the result from Theorem 7.3 corresponding to $n = 2$, so for convenience we record these here.

The Laplace transform of first and second derivatives

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0). \tag{9a}$$

$$\mathcal{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0). \tag{9b}$$

THEOREM 7.4

Transform of f' when f is discontinuous at $t = a$ Let $f(t)$ be continuous on $0 \le t < a$ and on $a < t < \infty$, and let it have a simple jump discontinuity at $t = a$ with the value $f_-(a)$ to the immediate left of a at $t = a-$ and the value $f_+(a)$ to the immediate right of $t = a$ at $a+$. Then if $\mathcal{L}\{f(t)\} = F(s)$,

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) + [f_-(a) - f_+(a)]e^{-as}.$$

Proof Using integration by parts, as in Theorem 7.2, we have

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= \int_0^{a-} e^{-st} f'(t)dt + \lim_{k\to\infty} \int_{a+}^{\infty} e^{-st} f'(t)dt \\ &= [e^{-st} f(t)]_0^{a-} + \lim_{k\to\infty} [e^{-sk} f(k) - e^{-as} f_+(a)] + sF(s) \\ &= sF(s) - f(0) + [f_-(a) - f_+(a)]e^{-as}.\end{aligned}$$

■

The next example illustrates the application of results (8) and (9) to a simple initial value problem.

EXAMPLE 7.10

Solve the initial value problem

$$y'' + 3y' + 2y = \sin 2t, \quad \text{where } y(0) = 2 \quad \text{and} \quad y'(0) = -1.$$

Solution Because of the linearity of the equation and of the Laplace transform operation, taking the Laplace transform of the differential equation we have

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\sin 2t\}.$$

Setting $\mathcal{L}\{y(t)\} = Y(s)$, and using the initial conditions $y(0) = 2$ and $y'(0) = -1$, we find from (9a,b) that

$$\mathcal{L}\{y''\} = s^2Y(s) - 2s + 1,$$

and

$$\mathcal{L}\{y'\} = sY(s) - 2.$$

Entry 9 in Table 7.1 shows that $\mathcal{L}\{\sin 2t\} = 2/(s^2+4)$, so combining these results enables the transformed differential equation to be written

$$s^2Y(s) - 2s + 1 + 3[sY(s) - 2] + 2Y(s) = \frac{2}{s^2+4},$$

or as

$$(s^2 + 3s + 2)Y(s) = \frac{2s^3 + 5s^2 + 8s + 22}{s^2 + 4}.$$

Solving for the Laplace transform of the solution gives

$$Y(s) = \frac{2s^3 + 5s^2 + 8s + 22}{(s^2+4)(s^2+3s+2)}.$$

When expressed in partial fraction form, $Y(s)$ becomes

$$Y(s) = \frac{-5}{4}\frac{1}{s+2} + \frac{17}{5}\frac{1}{s+1} - \frac{1}{20}\frac{2}{s^2+4} - \frac{3}{20}\frac{s}{s^2+4}.$$

Using the linearity property when taking the inverse Laplace transform, we have

$$\mathcal{L}^{-1}\{Y(s)\} = -\frac{5}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{17}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{20}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} - \frac{3}{20}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\},$$

so using Table 7.1 to identify the four transforms involved shows that the solution of the initial value problem is

$$y(t) = -\frac{5}{4}e^{-2t} + \frac{17}{5}e^{-t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t, \quad \text{for } t > 0.$$ ■

This example illustrates a fundamental difference between the solution of an initial value problem obtained by using the Laplace transform and that obtained by the previous methods that have been developed. In the other methods, when solving an initial value problem, first a general solution was found, and then the arbitrary constants were matched to the initial conditions. However, in the Laplace transform approach the initial conditions are incorporated when the equation is transformed, so the inversion of $Y(s)$ gives the required solution of the initial value problem immediately.

As the *structure* of the solution in Example 7.10 is typical of the structure obtained when solving all initial value problems for ordinary differential equations by means of the Laplace transform, a closer examination of it will help understand how the solution is generated.

Returning to the point where the equation was transformed, the result can be rewritten as

$$\underbrace{(s^2+3s+2)}_{\substack{\text{Transformed homogeneous equation}\\ \text{with } y'',\, y',\text{ and } y \text{ replaced, respectively,}\\ \text{by } s^2,\, s, \text{ and } 1}} Y(s) = \underbrace{2s+5}_{\substack{\text{Transformed initial}\\ \text{conditions}}} + \underbrace{\frac{2}{s^2+2}}_{\substack{\text{Transformed nonhomogeneous}\\ \text{term}}}$$

Setting $G(s) = 1/(s^2+3s+2)$, and denoting the transformed initial conditions by $I(s)$ and the transformed nonhomogeneous term by $R(s)$, the above result can be solved for $Y(s)$ and written in the form

$$Y(s) = G(s)I(s) + G(s)R(s). \tag{10}$$

transfer function

This shows how the transform $G(s)$, called in engineering applications the **transfer function** associated with the differential equation, modifies the transform of the initial conditions and the transform of the nonhomogeneous term to arrive at the transform $Y(s)$ of the solution. The name *transfer function* comes from the fact that when all the initial conditions are zero, so $I(s) = 0$, the only term generating a solution is the forcing function (the nonhomogeneous term), so (10) describes how the effect of the *input* is *transferred* to the *output* (the solution). In terms of Example 7.10 we can write

$$G(s) = \frac{Y(s)}{R(s)} = \frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{\sin 2t\}} = \frac{\mathcal{L}\{output\}}{\mathcal{L}\{input\}}. \tag{11}$$

In control theory the transfer function of a system characterizes the behavior of the entire system.

We now develop the most important operational properties of the Laplace transform, starting with the first shift theorem, also called the s-shift theorem.

THEOREM 7.5

the s-shift theorem

The first shift theorem or the s-shift theorem Let $\mathcal{L}\{f(t)\} = F(s)$ for $s > \gamma$. Then the Laplace transform of $e^{at}f(t)$ is obtained from $F(s)$ by replacing s by $s-a$, where $s - a > \gamma$. Thus,

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a) \quad \text{for } s-a>\gamma.$$

Conversely, the inverse transform

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t).$$

Proof From the conditions of the theorem, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$ for $s > \gamma$, so

$$\mathcal{L}\{e^{-at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)dt = \int_0^\infty e^{-(s-a)t}f(t)dt = F(s-a) \quad \text{for } s-a>\gamma.$$

The converse result follows by reversing this argument to arrive at the result

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t). \qquad \blacksquare$$

EXAMPLE 7.11

Use Theorem 7.5 to find $\mathcal{L}\{e^{at}t^n\}$, $\mathcal{L}\{e^{at}\cos bt\}$, and $\mathcal{L}\{e^{at}t\sin bt\}$.

Solution Using the Laplace transforms of t^n, $\cos bt$, and $t\sin bt$ listed as entries 3, 10, and 11 in Table 7.1, with a replaced by b in entries 10 and 11, and then replacing s by $s-a$ we find that

$$\mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}} \quad \text{for } s>0, \quad \mathcal{L}\{e^{at}\cos bt\} = \frac{(s-a)}{[(s-a)^2+b^2]} \quad \text{for } s>a,$$

and

$$\mathcal{L}\{e^{at}t\sin bt\} = \frac{2b(s-a)}{[(s-a)^2+b^2]^2} \quad \text{for } s>a. \qquad \blacksquare$$

EXAMPLE 7.12

Use Theorem 7.5 to find $\mathcal{L}^{-1}\{1/(s^2+4s+13)\}$.

Solution Completing the square in the denominator we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+13}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+3^2}\right\}.$$

A comparison with entry 13 in Table 7.1 shows that

$$\mathcal{L}^{-1}\{1/(s^2+4s+13)\} = \frac{1}{3}e^{-2t}\sin 3t. \qquad \blacksquare$$

We now derive the second shift theorem, also called the t-shift theorem, in which use will be made of the unit step function $H(t-a)$.

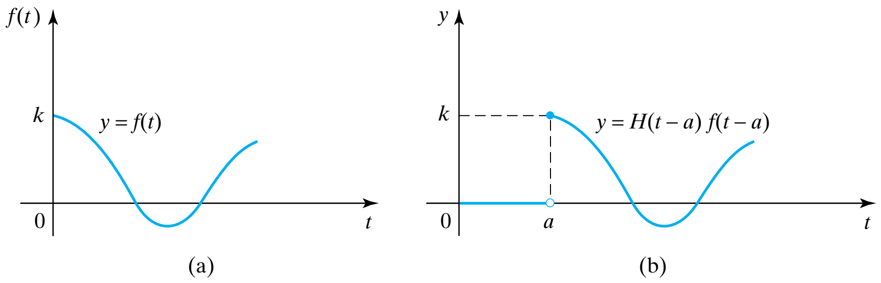


FIGURE 7.10 The relationship between $f(t)$ and $H(t-a)f(t-a)$.

THEOREM 7.6

the *t*-shift theorem

The second shift theorem or the *t*-shift theorem Let $\mathcal{L}\{f(t)\} = F(s)$. Then

$$\mathcal{L}\{H(t-a)f(t-a)\} = e^{-as}F(s)$$

and, conversely,

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = H(t-a)f(t-a).$$

Proof Before proving the theorem it is necessary to understand the precise meaning of $H(t-a)f(t-a)$. This can be seen by examining Fig. 7.10. The unit step function $H(t-a)$ is zero until $t = a$, when it jumps to the value 1 and thereafter remains constant for $t > a$. The function $f(t-a)$ is simply the function $f(t)$ with its origin shifted to $t = a$, so it can be considered to be the function $f(t)$ translated to the right by an amount a. Thus, $H(t-a)f(t-a)$ is a function that is zero until $t = a$, after which it reproduces the function $f(t)$ translated to the right by an amount a.

The result of the theorem is obtained as follows:

$$\mathcal{L}\{H(t-a)f(t-a)\} = \int_0^\infty e^{-st}H(t-a)f(t-a)dt = \int_a^\infty e^{-st}f(t-a)dt.$$

If we make the change of variable $\tau = t - a$, this becomes

$$\mathcal{L}\{H(t-a)f(t-a)\} = e^{-as}\int_0^\infty e^{-s\tau}f(\tau)d\tau$$

and so

$$\mathcal{L}\{H(t-a)f(t-a)\} = e^{-as}F(s).$$

The converse result follows by reversing this argument. ■

EXAMPLE 7.13

Use Theorem 7.6 to find (a) $\mathcal{L}\{H(t-4)\sin(t-4)\}$, (b) to show that $\mathcal{L}\{H(t-a)\} = e^{-as}/s$ in agreement with entry 7 in Table 7.1, and (c) to find $\mathcal{L}^{-1}\{se^{-as}/(s^2+b^2)\}$.

Solution **(a)** From entry 9 in Table 7.1 we have $\mathcal{L}\{\sin t\} = 1/(s^2+1)$, so applying Theorem 7.6 with $a = 4$ gives

$$\mathcal{L}\{H(t-4)\sin(t-4)\} = e^{-4s}/(s^2+1).$$

(b) Setting $f(t) = 1$ in Theorem 7.6 and using the fact that $\mathcal{L}\{1\} = 1/s$ gives

$$\mathcal{L}\{H(t-a)\} = e^{-as}/s.$$

(c) Entry 10 in Table 7.1 shows that $\mathcal{L}\{\cos bt\} = s/(s^2 + b^2)$, so using this in Theorem 7.6 gives

$$\mathcal{L}^{-1}\{se^{-as}/(s^2 + b^2)\} = H(t - a)\cos[b(t - a)].$$

The next example makes use of Theorem 7.6 when solving an initial value problem.

EXAMPLE 7.14

Solve the initial value problem

$$y'' + 3y' + 2y = H(t - \pi)\sin 2t \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 0.$$

Solution Setting $\mathcal{L}\{y(t)\} = Y(s)$, transforming the differential equation, and incorporating the initial conditions as in Example 7.10 gives

$$s^2 Y(s) - s + 3(sY(s) - 1) + 2Y(s) = \frac{2e^{-\pi s}}{s^2 + 4},$$

or

$$(s^2 + 3s + 2)Y(s) = s + 3 + \frac{2e^{-\pi s}}{s^2 + 4}.$$

As $s^2 + 3s + 2 = (s + 1)(s + 2)$, this last result can be written in the form

$$Y(s) = \frac{s + 3}{(s + 1)(s + 2)} + \frac{2e^{-\pi s}}{(s^2 + 4)(s + 1)(s + 2)}.$$

It is now necessary to invert $Y(s)$, and to accomplish this some algebraic manipulation will be necessary if we are to identify terms on the right with entries in Table 7.1. When expressed in terms of partial fractions, after a little manipulation $Y(s)$ becomes

$$Y(s) = \frac{2}{s + 1} - \frac{1}{s + 2} + e^{-\pi s}\left(\frac{2}{5}\frac{1}{s + 1} - \frac{1}{4}\frac{1}{s + 2} - \frac{1}{20}\frac{2}{s^2 + 4} - \frac{3}{20}\frac{s}{s^2 + 4}\right).$$

Each term can now be identified as the transform of an entry in Table 7.1, though as the last four terms are multiplied by $e^{-\pi s}$ their inverse Laplace transforms will need to be obtained by using Theorem 7.6. As a result, $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ becomes

$$\begin{aligned} y(t) &= 2e^{-t} - e^{-2t} + H(t - \pi) \\ &\quad \times \left(\frac{2}{5}e^{-(t-\pi)} - \frac{1}{4}e^{-2(t-\pi)} - \frac{1}{20}\sin 2(t - \pi) - \frac{3}{20}\cos 2(t - \pi)\right), \end{aligned}$$

for $t > 0$. A graph of this solution is shown in Fig. 7.11, from which it can be seen that in the interval $0 < t < \pi$ the solution $y(t)$ only involves the first two terms, and so decays exponentially. At $t = \pi$ the forcing function $\sin 2t$ is switched on, after which all the exponential terms decay to zero as $t \to \infty$, leaving only the periodic steady state solution.

THEOREM 7.7

differentiating a transform

Differentiation of a transform: Multiplication of $f(t)$ by t^n Let $\mathcal{L}\{f(t)\} = F(s)$. Then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}.$$

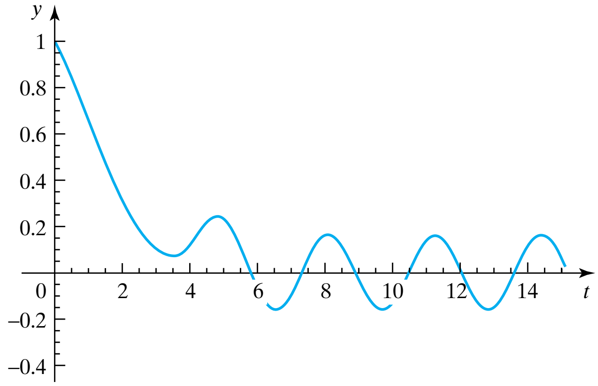


FIGURE 7.11 The solution $y(t)$ showing the influence of the forcing function after $t = \pi$.

Proof By definition

$$\int_0^\infty e^{-st} f(t)dt = F(s),$$

so differentiating under the integral sign with respect to s gives

$$\frac{dF(s)}{ds} = \int_0^\infty \frac{\partial(e^{-st})}{\partial s} f(t)dt,$$

and so

$$\frac{dF(s)}{ds} = \int_0^\infty (-t)e^{-st} f(t)dt = -\int_0^\infty e^{-st} t f(t)dt,$$

which is the result of the theorem when $n = 1$. Each subsequent differentiation will introduce a further factor $(-t)$ into the integrand, leading the general result of the theorem. ■

EXAMPLE 7.15

Use Theorem 7.7 to find (a) $\mathcal{L}\{t \sin at\}$ and (b) $\mathcal{L}\{t\, e^{at} \cos bt\}$.

Solution **(a)** Entry 9 in Table 7.1 shows that $\mathcal{L}\{\sin at\} = a/(s^2 + a^2)$ for $s > 0$, so from Theorem 7.7

$$\mathcal{L}\{t \sin at\} = (-1)\frac{d}{ds}\frac{a}{(s^2 + a^2)} = \frac{2as}{(s^2 + a^2)^2} \quad \text{for } s > 0,$$

in agreement with entry 11 in Table 7.1.
(b) Entry 14 in Table 7.1 shows that $\mathcal{L}\{e^{at} \cos bt\} = (s - a)/[(s - a)^2 + b^2]$ for $s > a$, so from Theorem 7.7

$$\mathcal{L}\{t\, e^{at} \cos bt\} = (-1)\frac{d}{ds}\frac{(s - a)}{[(s - a)^2 + b^2]}$$
$$= \frac{(s - a)^2 - b^2}{[(s - a)^2 + b^2]^2} \quad \text{for } s > a.$$

These examples show that, in many cases, less effort is involved finding transforms by means of Theorem 7.7 than by direct use of the definition of the Laplace transform. ■

THEOREM 7.8 **scaling a transform**

Scaling theorem Let $\mathcal{L}\{f(t)\} = F(s)$. Then if $k > 0$,

$$\mathcal{L}\{f(kt)\} = \frac{1}{k}F\left(\frac{s}{k}\right).$$

Proof The result follows by setting $u = kt$ in the definition of the Laplace transform, because

$$\begin{aligned}\{f(kt)\} &= \int_0^\infty e^{-st} f(kt)dt \\ &= \frac{1}{k}\int_0^\infty e^{-s(u/k)} f(u)du \\ &= \frac{1}{k}\int_0^\infty e^{-(s/k)u} du \\ &= \frac{1}{k}F\left(\frac{s}{k}\right).\end{aligned}$$

■

EXAMPLE 7.16 If $\mathcal{L}\{f(t)\} = e^{-3s}(1-2s)/(2s^2 - s + 1)$, find $\{f(3t)\}$.

Solution In this case $k = 3 > 0$, so from Theorem 7.8, replacing s by $s/3$ in $\mathcal{L}\{f(t)\}$ and multiplying the result by 1/3 gives

$$\begin{aligned}\mathcal{L}\{f(3t)\} &= \frac{1}{3}\frac{e^{-s}(1-2s/3)}{(2(s/3)^2 - s/3 + 1)} \\ &= \frac{e^{-s}(3-2s)}{2s^2 - 3s + 9}.\end{aligned}$$

■

Many functions whose Laplace transform is required are periodic functions with period T, though they are not necessarily continuous functions for all $t > 0$. In the Laplace transform, where only the behavior of a function $f(t)$ for $t > 0$ is involved, a **periodic function** with **period** T is defined as a function $f(t)$ with the property that T is the smallest value for which

$$f(t+T) = f(t) \quad \text{for all } t > 0. \tag{12}$$

An example of a piecewise continuous function $f(t)$ with period T that is defined for $t > 0$ is shown in Fig. 7.12.

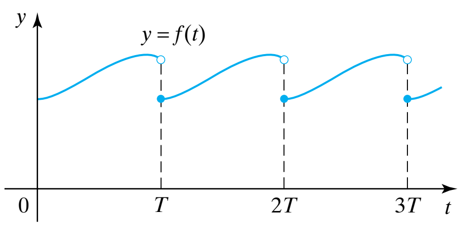


FIGURE 7.12 A function $f(t)$ with period T.

THEOREM 7.9

transforming a periodic function

Transform of a periodic function with period T Let $f(t)$ be a periodic function with period T such that $\int_0^T e^{-st} f(t)dt$ is finite. Then

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)dt \quad \text{for } s > 0.$$

Proof In the definition of the Laplace transform we divide the interval of integration into subintervals of length T and write

$$\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)dt + \int_T^{2T} e^{-st} f(t)dt + \cdots.$$

Then, because of the periodicity of $f(t)$, the function $f(t)$ will be the same in each integral. Consequently, changing the variable in the $(r+1)$th integral to $t = \tau + rT$ with $r = 0, 1, 2, \ldots$ gives

$$\begin{aligned}\int_0^T e^{-s(\tau+rT)} f(\tau)d\tau &= e^{-rsT} \int_0^T e^{-s\tau} f(\tau)d\tau \quad \text{for } r = 0,\ 1, 2, \ldots\\ &= e^{-rsT} \int_0^T e^{-st} f(t)dt,\end{aligned}$$

where the dummy variable τ has been replaced by t. Substituting this result into the original integral gives

$$\mathcal{L}\{f(t)\} = [1 + e^{-Ts} + e^{-2Ts} + \cdots] \int_0^T e^{-st} f(t)dt,$$

which is finite because we have assumed that $\int_0^T e^{-st} f(t)dt$ is finite. The bracketed terms form a geometrical series with the common ratio $e^{-Ts} < 1$, so its sum is $1/(1 - e^{-Ts})$, and thus

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)dt, \quad \text{for } s > 0,$$

and the proof is complete. ■

The necessity of the condition in Theorem 7.9 that $\int_0^T e^{-st} f(t)dt$ is finite arises because periodic functions exist for which this integral is divergent.

EXAMPLE 7.17

Find the Laplace transform of the square wave shown in Fig. 7.13.

Solution As the function is discontinuous with period $2a$ we compute the integral in Theorem 7.9 in two parts as

$$\begin{aligned}\int_0^{2a} e^{-st} f(t)dt &= \int_0^a ke^{-st}dt + \int_a^{2a} (-k)e^{-st}dt\\ &= \frac{k}{s}(1 - e^{-as}) + \frac{k}{5}(e^{-2as} - e^{-as})\\ &= \frac{k}{s}(1 + e^{-2as} - 2e^{-as}).\end{aligned}$$

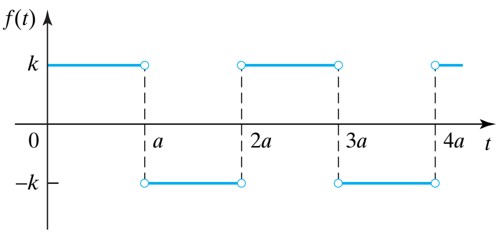


FIGURE 7.13 A square wave with period $2a$.

Then from Theorem 7.9 we have

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \frac{k(1+e^{-2as}-2e^{-as})}{s(1-e^{-2as})}\\ &= \frac{k(1-e^{-as})}{s(1+e^{-as})}\\ &= \frac{k(e^{as/2}-e^{-as/2})}{s(e^{as/2}+e^{-as/2})}\\ &= \frac{k\sinh(as/2)}{s\cosh(as/2)} = \frac{k}{s}\tanh(as/2) \quad \text{for } s>0.\end{aligned}$$

EXAMPLE 7.18 Use Theorem 7.9 to show that $\mathcal{L}\{\sin t\} = 1/(s^2+1)$ and Theorem 7.8 to show that $\mathcal{L}\{\sin at\} = a/(s^2+a^2)$.

Solution The function $f(t) = \sin t$ is periodic with period 2π and $\int_0^{2\pi} e^{-st}\sin t\,dt$ is finite, so from Theorem 7.9 we have

$$\begin{aligned}\mathcal{L}\{\sin t\} &= \frac{1}{(1-e^{-2\pi s})}\int_0^{2\pi} e^{-st}\sin t\,dt\\ &= \frac{1}{(1-e^{-2\pi s})}\left(\frac{1}{s^2+1}-\frac{e^{-2\pi s}}{s^2+1}\right)\\ &= \frac{1}{s^2+1} \quad \text{for } s>0.\end{aligned}$$

Setting $k=a$ in Theorem 7.8 and using the preceding result gives

$$\begin{aligned}\mathcal{L}\{\sin at\} &= \frac{1}{a}\frac{1}{[(s/a)^2+1]}\\ &= \frac{a}{s^2+a^2} \quad \text{for } s>0.\end{aligned}$$

EXAMPLE 7.19 Find the Laplace transform of the solution of the initial value problem

$$y''+3y'+2y = f(t), \quad \text{where } y(0)=y'(0)=0$$

and $f(t)$ is the square wave in Example 7.17.

Solution Transforming the equation as in Examples 7.10 and 7.14 and using the result of Example 7.17 gives

$$s^2Y(s)+3sY(s)+2Y(s) = \frac{k}{s}\tanh(as/2),$$

so

$$Y(s) = \frac{k\tanh(as/2)}{s(s^2+3s+2)}.$$

The convolution operation

Let the functions $f(t)$ and $g(t)$ be defined for $t \geq 0$. Then the **convolution** of the functions f and g denoted by $(f * g)(t)$, and in abbreviated form by $(f * g)$, is defined as the integral

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

convolution and the convolution theorem

The change of variable $v = t - \tau$ followed by the replacement of the dummy variable v by t shows that the convolution operation is *commutative*, so

$$(f * g)(t) = (g * f)(t). \tag{13}$$

EXAMPLE 7.20

Find $(t^2 * \cos t)$ and $(\cos t * t^2)$ and hence confirm the equality of these two convolution operations. Compare the effort required in each case.

Solution We have

$$\begin{aligned}(t^2 * \cos t) &= \int_0^t \tau^2 \cos(t-\tau)d\tau \\ &= \int_0^t \tau^2[\cos t\cos\tau + \sin t\sin\tau]d\tau \\ &= \cos t\int_0^t \tau^2\cos\tau\, d\tau + \sin t\int_0^t \tau^2 \sin\tau\, d\tau \\ &= 2(t-\sin t).\end{aligned}$$

Similarly,

$$\begin{aligned}(\cos t * t^2) &= \int_0^t \cos\tau(t-\tau)^2 d\tau \\ &= t^2\int_0^t \cos\tau\, d\tau - 2t\int_0^t \tau\cos\tau\, d\tau + \int_0^t \tau^2\cos\tau\, d\tau \\ &= 2(t-\sin t).\end{aligned}$$

While confirming that the convolution operation is commutative, this example also shows that sometimes calculating $(f * g)(t)$ is simpler than calculating $(g * f)(t)$.

The convolution operation has various uses, one of the most important of which occurs in the following important theorem that expresses the relationship between the product of two Laplace transforms $F(s)$ and $G(s)$ and the convolution of their transform pairs $f(t)$ and $g(t)$.

THEOREM 7.10

The convolution theorem Let $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$. Then

$$\mathcal{L}\{(f * g)(t)\} = F(s)G(s)$$

or, equivalently,

$$\mathcal{L}\left\{\int_0^t f(\tau)g(t-\tau)d\tau\right\} = F(s)G(s).$$

Conversely,

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Proof From the definition of the Laplace transform and the convolution operation, we have

$$\mathcal{L}\{(f * g)(t)\} = \int_0^\infty e^{-st}\left[\int_0^t f(\tau)g(t-\tau)d\tau\right]dt.$$

Inspection of Fig. 7.14 shows that interchanging the order of integration allows the integral to be written as

$$\mathcal{L}\{(f * g)(t)\} = \int_0^\infty f(\tau)\left[\int_\tau^\infty e^{-st}g(t-\tau)dt\right]d\tau.$$

Using the second shift theorem reduces the inner integral to $e^{-st}G(s)$, so that

$$\begin{aligned}\mathcal{L}\{(f * g)(t)\} &= \int_0^\infty G(s)e^{-s\tau}f(\tau)d\tau \\ &= G(s)\int_0^\infty e^{-s\tau}f(\tau)d\tau \\ &= G(s)F(s).\end{aligned}$$

The converse result follows if we reverse the argument to find the inverse Laplace transform of $F(s)G(s)$. ■

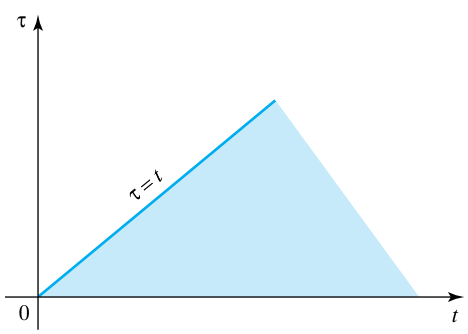


FIGURE 7.14 Region of integration for Theorem 7.10.

EXAMPLE 7.21 Use Theorem 7.10 to find (a) $\mathcal{L}\{t^2 * \cos t\}$ and (b) $\mathcal{L}^{-1}\{s/(s^2+a^2)^2\}$.

Solution

(a) $\mathcal{L}\{t^2\} = 2/s^3$ and $\mathcal{L}\{\cos t\} = s/(s^2+a^2)$, so from Theorem 7.10

$$\mathcal{L}\{t^2 * \cos t\} = \mathcal{L}\{t^2\}\,\mathcal{L}\{\cos t\} = \frac{2s}{(s^2+a^2)}.$$

(b) Writing

$$\frac{s}{(s^2+a^2)^2} = \frac{1}{(s^2+a^2)}\frac{s}{(s^2+a^2)}$$

shows that in Theorem 7.10 we may take

$$F(s) = \frac{1}{(s^2+a^2)} \quad \text{and} \quad G(s) = \frac{s}{(s^2+a^2)}.$$

So as $\mathcal{L}^{-1}\{F(s)\} = (1/a)\sin at$ and $\mathcal{L}^{-1}\{G(s)\} = \cos at$, it follows from Theorem 7.10 that

$$\begin{aligned}\mathcal{L}^{-1}\{s/(s^2+a^2)^2\} &= (1/a)(\sin at * \cos at)\\ &= \frac{1}{a}\int_0^t \sin a\tau \cos a(t-\tau)d\tau\\ &= \frac{1}{2a}t\sin at,\end{aligned}$$

in agreement with entry 11 in Table 7.1. ■

When evaluating convolution integrals of this type, instead of expanding a term such as $\cos a(t-\tau)$ and $\sin a(t-\tau)$ using integration by parts, it is often quicker to replace $\sin at$ and $\cos at$ by

$$\sin at = (e^{iat} - e^{-iat})/(2i) \quad \text{and} \quad \cos a(t-\tau) = \left(e^{i(t-\tau)} + e^{-i(t-\tau)}\right)/2$$

before performing the integrations, and again using these identities to interpret the result in terms of trigonometric functions.

EXAMPLE 7.22 Solve the initial value problem

$$y'' + 4y' + 13y = 2e^{-2t}\sin 3t \quad \text{with } y(0) = 1 \quad \text{and} \quad y'(0) = 0.$$

Solution Before we solve this initial value problem, it should be noted that the complementary function is

$$y_c(t) = e^{-2t}(C_1\cos 3t + C_2\sin 3t),$$

so the nonhomogeneous term $2e^{-2t}\sin 3t$ is contained in $y_c(t)$. It will be seen that, unlike the special cases that arise when determining a particular integral by the method of undetermined coefficients, this situation does not give rise to a special case when the solution is obtained by means of the Laplace transform.

Transforming the equation in the usual way gives

$$s^2Y(s) - s + 4(sY(s) - 1) + 13Y(s) = \frac{6}{s^2+4s+13},$$

and so

$$Y(s) = \frac{s+4}{s^2+4s+13} + \frac{6}{(s^2+4s+13)^2}.$$

Writing $s + 4 = s + 2 + (2/3)3$ allows $Y(s)$ to be rewritten as

$$Y(s) = \frac{s+2}{(s+2)^2+3^2} + \frac{2}{3}\frac{3}{(s+2)^2+3^2} + \frac{6}{[(s+2)^2+3^2]^2}.$$

Taking the inverse Laplace transform of $Y(s)$ and using entries 13 and 14 of Table 7.1 leads to the result

$$y(t) = e^{-2t}\left[\cos 3t + \frac{2}{3}\sin 3t\right] + \mathcal{L}^{-1}\{6/[(s+2)^2+3^2]^2\}.$$

To find $\mathcal{L}^{-1}\{6/[(s+2)^2+3^2]^2\}$, we first write this as

$$\frac{6}{[(s+2)^2+3^2]^2} = \frac{2}{3}\left(\frac{3}{(s+2)^2+3^2}\right)\left(\frac{3}{(s+2)^2+3^2}\right),$$

and then, from entry 13 in Table 7.1, we find that $\mathcal{L}^{-1}\{3/[(s+2)^2+3^2]\} = e^{-2t}\sin 3t$. An application of Theorem 7.10 shows that

$$\begin{aligned}
\mathcal{L}^{-1}\{6/[(s+2)^2+3^2]^2\} &= \frac{2}{3}(e^{-2t}\sin 3t * e^{-2t}\sin 3t)\\
&= \frac{2}{3}\int_0^t e^{-2\tau}\sin 3\tau e^{-2(t-\tau)}\sin 3(t-\tau)d\tau\\
&= \frac{2}{3}e^{-2t}\int_0^t \sin 3\tau \sin 3(t-\tau)d\tau\\
&= \frac{2}{3}e^{-2t}\left(\frac{1}{6}\sin 3t - \frac{1}{2}t\cos 3t\right).
\end{aligned}$$

Substituting this result in the expression for $y(t)$ shows that the solution of the initial value problem is

$$y(t) = e^{-2t}\left(\cos 3t + \frac{7}{9}\sin 3t - \frac{1}{3}t\cos 3t\right), \quad \text{for } t > 0. \qquad ■$$

Although the previous example could have been solved by the method of undetermined coefficients, the next two examples cannot be solved in this manner. The first involves a special type of equation called an **integral equation**, and the second an **integro-differential** equation.

integral equation

An equation of the form

$$y(t) = f(t) + \lambda\int_0^t K(t,\tau)y(\tau)d\tau \tag{14}$$

is called a **Volterra integral equation**, where λ is a parameter and $K(t,\ \tau)$ is called the **kernel** of the integral equation. Equations of this type are often associated with the solution of initial value problems. The Laplace transform is well suited to the solution of such integral equations when the kernel $K(t,\tau)$ has a special form that depends on t and τ only through the difference $t-\tau$, because then $K(t,\tau) = K(t-\tau)$ and the integral in (14) becomes a convolution integral.

An examination of the Volterra integral equation in (14) shows it to be essentially the integral form of an initial value problem, and it relates the solution $y(t)$ at the current time t to an integral of the past history of the solution over the interval $[0, t]$.

The following is a simple example of a problem that leads to a Volterra integral equation. Determine the amount of a manufactured material contained in a store from time $t = 0$ until time t, if the only supply of material comes immediately from the manufacturer and it begins degrading exponentially with time from the moment it enters the store. Let the amount of material present at time $t = 0$ be Q and the amount present in the store at time t be $y(t)$, and suppose it degrades exponentially as e^{-kt} with $k > 0$. Then, by time t, the amount of material that entered the store at time τ but has not degraded is $e^{-k(t-\tau)}y(\tau)$. Thus the amount of material present at time t is determined by the solution of the Volterra integral equation

$$y(t) = Qe^{-kt} + \int_0^t e^{-k(t-\tau)}y(\tau)d\tau.$$

By using the method of solution explained in the next example, the solution of this problem is easily shown to be

$$y(t) = Qe^{-(k-1)t}.$$

EXAMPLE 7.23 Solve the Volterra integral equation

$$y(t) = 2e^{-t} + \int_0^t \sin(t-\tau)y(\tau)d\tau.$$

Solution The Laplace transform of the integral equation is

$$Y(s) = \frac{2}{s+1} + \mathcal{L}\int_0^t \sin(t-\tau)y(\tau)d\tau,$$

and after applying Theorem 7.10 to the last term the equation for $Y(s)$ becomes

$$Y(s) = \frac{2}{s+1} + \frac{Y(s)}{s^2+1}.$$

Solving for $Y(s)$ and expanding the result in partial fractions shows that

$$Y(s) = \frac{2(s^2+1)}{s^2(s+1)} = \frac{2}{s^2} - \frac{2}{s} + \frac{4}{s+1}.$$

Taking the inverse Laplace transform shows the solution to be

$$y(t) = 2t - 2 + 4e^{-t}, \quad \text{for } t > 0.$$ ■

The next example is a differential equation of an unusual type, because the function $y(t)$ occurs not only as the dependent variable in the differential equation, but also inside a convolution integral that forms the nonhomogeneous term. Equations of this type that involve both the integral of an unknown function and its derivative are called **integro-differential equations**. These equations occur in many applications of mathematics, one of which arises in the continuum mechanics of polymers, where the dynamical response $y(t)$ of certain types of material at time t depends on a derivative of $y(t)$ and the time-weighted cumulative effect of what has happened to the material prior to time t. For obvious reasons materials of this type are called *materials with memory*.

integro-differential equation

An example of an integro-differential equation was obtained in Section 5.3(d) when considering the R–L–C circuit in Fig. 5.4, though at the time this was not recognized. When the circuit was closed, and the charge q on the capacitor was allowed to flow causing a current $i(t)$ in the circuit, the equation determining $i(t)$ was shown to be

$$L\frac{di}{dt} + Ri + \frac{q}{C} = 0.$$

To recognize that this is an integro-differential equation, we use the result that at time t we have $q = \int_0^t i(\tau)d\tau$, so the equation determining $i(t)$ becomes the integro-differential equation

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i(\tau)d\tau.$$

In this case it was possible to reduce this to a second order constant coefficient differential equation for $i(t)$, but in other more complicated cases a reduction of this type may not be possible.

EXAMPLE 7.24

Solve the equation

$$y'' + y = \int_0^t \sin\tau\, y(t-\tau)d\tau,$$

subject to the initial conditions $y(0) = 1$ and $y'(0) = 0$.

Solution Taking the Laplace transform in the usual way gives

$$s^2Y(s) - s + Y(s) = \mathcal{L}\int_0^t \sin\tau\, y(t-\tau)d\tau.$$

The last term is the Laplace transform of a convolution integral, so from Theorem 7.10 it follows that

$$\begin{aligned}\mathcal{L}\left\{\int_0^t \sin\tau\, y(t-\tau)d\tau\right\} &= \mathcal{L}\{\sin t\}\mathcal{L}\{y(t)\}\\ &= \frac{Y(s)}{s^2+1}.\end{aligned}$$

Using this result in the transformed equation, solving for $Y(s)$, and expanding the result using partial fractions gives

$$Y(s) = \frac{s^2+1}{s(s^2+2)} = \frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{s}{(s^2+2)}.$$

After the inverse Laplace transform is taken, the solution becomes

$$y(t) = \frac{1}{2}(1+\cos\sqrt{2}t), \quad \text{for } t > 0.$$

■

THEOREM 7.11

transforming an integral

The transform of an integral Let $f(t)$ be a piecewise continuous function such that $|f(t)| \leq Me^{kt}$ for $k > 0$ and all $t \geq 0$. Then, if $\mathcal{L}\{f(t)\} = F(s)$,

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s} \quad \text{for} \quad s > k,$$

and, conversely,

$$\mathcal{L}^{-1}\{F(s)/s\} = \int_0^t f(\tau)d\tau.$$

Proof The condition $|f(t)| \le Me^{kt}$ is sufficient to ensure the existence of the Laplace transform $F(s)$, so writing $h(t) = \int_0^t f(\tau)d\tau$ we have

$$|h(t)| \le \int_0^t |f(\tau)|d\tau \le M\int_0^t e^{k\tau}d\tau \le M\frac{e^{kt}}{k} \quad \text{for} \quad t \ge 0.$$

This result shows that $|h(t)|$ grows no faster than $|f(t)|$ as $t \to \infty$, so the existence of the Laplace transform $Y(s)$ ensures the existence of the Laplace transform of $h(t)$. Using the fundamental result from the calculus that $h'(t) = f(t)$ together with Theorem 7.2 means that, apart from points where $f(t)$ is discontinuous,

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{h'(t)\} = s\mathcal{L}\{h(t)\} = s\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\},$$

and so

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}.$$

The converse result follows by taking the inverse Laplace transform and the proof is complete. ■

EXAMPLE 7.25 Find (a) $\mathcal{L}\{\int_0^t \tau\cos a\tau d\tau\}$ and (b) $\mathcal{L}^{-1}\{1/[s(s^2+a^2)]\}$.

Solution **(a)** As $\mathcal{L}\{t\cos at\} = (s^2-a^2)/(s^2+a^2)^2$ for $s > 0$, an application of Theorem 7.11 shows that

$$\mathcal{L}\left\{\int_0^t \tau\cos a\tau d\tau\right\} = \frac{s^2-a^2}{s(s^2+a^2)^2} \quad \text{for } s > 0.$$

(b) We can write

$$\frac{1}{s(s^2+a^2)} = \frac{1}{s^2+a^2}\frac{1}{s}.$$

So if we set $F(s) = 1/(s^2+a^2)$, for which $f(t) = \mathcal{L}^{-1}F(s) = (1/a)\sin at$, it follows from Theorem 7.11 that

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \int_0^t \frac{1}{a}\sin a\tau d\tau$$
$$= \frac{1}{a^2}(1-\cos at),$$

in agreement with entry 17 of Table 7.1. ■

THEOREM 7.12

integrating a transform

The integral of a transform Let $f(t)/t$ be piecewise continuous, defined for $t \geq 0$ and such that $|f(t)/t| \leq Me^{-kt}$ for $t \geq 0$. Then if $\mathcal{L}\{f(t)/t\} = G(s)$ for $s > k$, and $\mathcal{L}\{f(t)\} = F(s)$,

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(u)du$$

and, conversely,

$$\mathcal{L}^{-1}\{G(s)\} = \frac{-1}{t}\mathcal{L}^{-1}\{G'(s)\}.$$

Proof We have

$$G(s) = \int_0^{\infty} e^{-st}\frac{f(t)}{t} \quad \text{for} \quad s > k.$$

However, from Theorem 7.7,

$$G'(s) = \int_0^{\infty} e^{-st}(-t)\frac{f(t)}{t}dt = -\int_0^{\infty} e^{-st} f(t)dt = -F(s),$$

so after integration we have

$$\int_s^{\infty} F(u)du = -\int_s^{\infty} G'(u)du = G(s) - G(\infty)$$

To proceed further we now make use of the fact that the condition $|f(t)/t| \leq Me^{-kt}$ implies that $G(s)_{\lim s\to\infty} = 0$, showing that

$$G(s) = \mathcal{L}\{f(t)/t\} = \int_s^{\infty} F(u)du \quad \text{for } s > k.$$

The converse result follows by taking the inverse Laplace transform and using the fact that $\mathcal{L}^{-1}\{G(s)\} = f(t)/t$ together with the result $\mathcal{L}\{f(t)\} = F(s) = -G'(s)$. ■

EXAMPLE 7.26

Find

$$\text{(a) } \mathcal{L}\left\{\frac{\sin at}{t}\right\} \quad \text{and} \quad \text{(b) } \mathcal{L}^{-1}\left\{\ln\left(\frac{s+a}{s+b}\right)\right\}.$$

Solution **(a)** The function $(\sin at)/t$ is defined and finite for all $t > 0$, so Theorem 7.12 can be applied. If we use the fact that $\mathcal{L}\{\sin at\} = a/(s^2 + a^2)$, it follows from the first part of Theorem 7.12 that

$$\begin{aligned}\mathcal{L}\left\{\frac{\sin at}{t}\right\} &= \int_s^{\infty} \frac{a}{u^2 + a^2}du \\ &= \pi/2 - \text{Arctan}\,(s/a) \\ &= \text{Arctan}\,(a/s).\end{aligned}$$

(b) If we set

$$G(s) = \ln\left(\frac{s+a}{s+b}\right),$$

differentiation gives

$$G'(s) = \frac{b-a}{(s+a)(s+b)} = \frac{1}{s+a} + \frac{1}{s+b},$$

from which we see that

$$\mathcal{L}^{-1}\{G'(s)\} = e^{-at} - e^{-bt}.$$

From the second part of Theorem 7.11 we have

$$\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\ln\left(\frac{s+a}{s+b}\right)\right\} = \frac{-1}{t}\mathcal{L}^{-1}\{G'(s)\}$$
$$= (e^{-bt} - e^{-at})/t.$$

The conditions of Theorem 7.11 assert that method used to derive this result is permissible if $\mathcal{L}^{-1}\{G(s)\}$ is defined and finite for $t \geq 0$. We see from the preceding result that $\mathcal{L}^{-1}\{G(s)\}$ is defined and finite for $t > 0$ and $\lim_{t\to 0}[(e^{-bt} - e^{-at})/t] = a - b$, so the conditions of the theorem are satisfied and we have shown that

$$\mathcal{L}^{-1}\left\{\ln\left(\frac{s+a}{s+b}\right)\right\} = (e^{-bt} - e^{-at})/t.$$ ■

The theorem that follows shows how the initial values $f(0), f'(0), \ldots,$ of a suitably differentiable function $f(t)$ can be found directly from its Laplace transform $F(s)$. An example of the use of the theorem is to be found in Section 7.3(d) when determining the Laplace transform of a function known only as the solution of a differential equation.

THEOREM 7.13

relating initial values and the transform

The initial value theorem Let $\mathcal{L}\{f(t)\} = F(s)$ be the Laplace transform of an n times differentiable function $f(t)$. Then

$$f^{(r)}(0) = \lim_{s\to\infty}\left\{s^{r+1}F(s) - s^r f(0) - s^{r-1}f'(0) - \cdots - sf^{(r-1)}(0)\right\},$$
$$r = 0, 1, \ldots, n.$$

In particular,

$$f(0) = \lim_{s\to\infty}\{sF(s)\}, \quad f'(0) = \lim_{s\to\infty}\{s^2F(s) - sf(0)\}$$
$$f''(0) = \lim_{s\to\infty}\{s^3F(s) - s^2f(0) - sf'(0)\}.$$

Proof The theorem follows directly from Theorem 7.3 by first replacing n by $r+1$ and rewriting the result as

$$f^{(r)}(0) = s^{r+1}F(s) - s^r f(0) - \cdots - sf^{(r-1)}(0) - \mathcal{L}\left\{f^{(r+1)}(t)\right\}.$$

Then, provided $f^{(r+1)}(t)$ satisfies the sufficiency condition for the existence of a Laplace transform given in (3), it follows that for some $M > 0$ and $k > 0$

$$\mathcal{L}\left\{f^{(r+1)}(t)\right\} < M/(s-k) \quad \text{for} \quad s > k \quad \text{and} \quad r = 0, 1, \ldots, n.$$

As a result,

$$\lim_{s\to\infty}\left\{f^{(r+1)}(t)\right\} = 0,$$

and the theorem is proved. ■

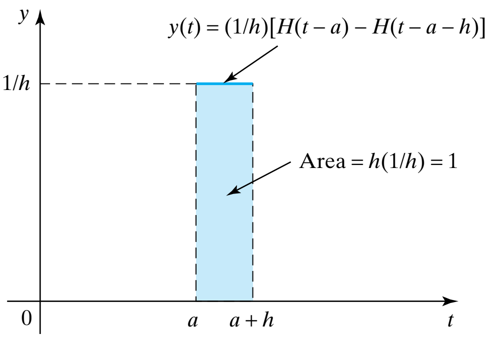


FIGURE 7.15 $\delta(t-a) = \lim_{h\to 0} y(t)$.

EXAMPLE 7.27 Given that $F(s) = 2as/(s^2+a^2)^2$, use Theorem 7.13 to find $f(0)$, $f'(0)$, and $f''(0)$. Use $f(t) = \mathcal{L}^{-1}\{F(s)\} = t \sin at$ to confirm the results by direct differentiation.

Solution From Theorem 7.13

$$f(0) = \lim_{s\to\infty}\{sF(s)\} = \lim_{s\to\infty} \frac{2as^2}{(s^2+a^2)^2} = 0,$$

$$f'(0) = \lim_{s\to\infty}\{s^2F(s) - sf(0)\} = \lim_{s\to\infty} \frac{2as^3}{(s^2+a^2)^2} = 0,$$

$$f''(0) = \lim_{s\to\infty}\{s^3F(s) - s^2f(0) - sf'(0)\} = \lim_{s\to\infty} \frac{2as^4}{(s^2+a^2)^2} = 2a.$$

These results are easily confirmed by differentiation of $f(t) = t \sin at$. ■

The last operational property to be considered concerns the **Dirac delta function**, usually abbreviated to the **delta function** and sometimes called the **unit impulse function**. The Dirac delta function, named after the Oxford University Nobel laureate mathematical physicist P. A. M. Dirac and denoted by $\delta(t-a)$, is actually a limiting mathematical *operation*, and not a function as its name implies. For our purposes the delta function can be considered to be the limit of a rectangular "pulse" of height h and width $1/h$ in the limit as $h \to \infty$. Thus the area of the graph representing the pulse remains constant at 1 as $h \to \infty$, while its height increases to infinity and its width decreases to zero. The graphical representation of such a pulse $f(t) = \delta(t-a)$ located at $t = a$, before proceeding to the limit, is shown in Fig. 7.15.

We adopt the following definition of the delta function in terms of the unit step function.

The delta function

the delta or impulse function

The **delta function** located at $t = a$ and denoted by $\delta(t-a)$ is defined as the limit

$$\delta(t-a) = \lim_{h\to 0} \frac{1}{h}[H(t-a) - H(t-a-h)].$$

The operational property of the delta function, usually called its **filtering property** and sometimes its **sifting property**, is represented by the following theorem.

THEOREM 7.14

a useful property of the delta function

Filtering property of the delta function Let $f(t)$ be defined and integrable over all intervals contained within $0 \le t < \infty$, and let it be continuous in a neighborhood of a. Then for $a \ge 0$

$$\int_0^\infty f(t)\delta(t-a)dt = f(a).$$

Proof From the definition of the delta function,

$$\int_0^\infty f(t)\delta(t-a)dt = \lim_{h\to 0}\int_a^{a+h}\frac{f(t)}{h}dt,$$

so applying the mean value theorem for integrals we have

$$\int_0^\infty f(t)\delta(t-a)dt = \lim_{h\to 0}\left[h\left(\frac{1}{h}\right)f(t_h)\right],$$

where $a < t_h < a+h$. In the limit as $h \to 0$ the variable $t_h \to a$, showing that

$$\int_0^\infty f(t)\delta(t-a)dt = f(a),$$

and the theorem is proved. ■

Consideration of the definition of the delta function suggests that, in a sense, $\delta(t-a)$ is the derivative of the unit step function $H(t-a)$, though the justification of this conjecture requires arguments involving **generalized functions** that are beyond the scope of this account.

In mechanical problems the delta function is used to represent an *impulse*, defined as the integral of a large force applied locally for a very short time. The delta function has many other applications, such as the distribution of point masses along a supporting beam, whereas in electrical systems it can be used to represent the brief application of a very large voltage, or the sudden discharge of energy contained in a capacitor.

A purely formal derivation of the Laplace transform of the delta function proceeds as follows. By definition,

$$\mathcal{L}\{\delta(t-a)\} = \int_0^\infty e^{-st}\delta(t-a)dt.$$

An application of the filtering property of Theorem 7.14 reduces this to

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}. \tag{15}$$

As a special case we have

$$\mathcal{L}\{\delta(t)\} = 1. \tag{16}$$

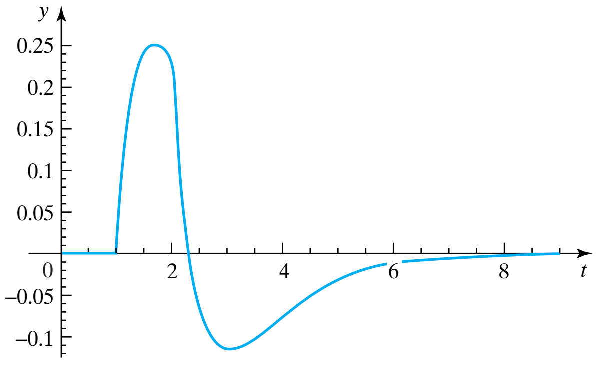


FIGURE 7.16 The solution $y(t)$ as a function of the time t.

EXAMPLE 7.28

Solve the initial value problem

$$y'' + 3y' + 2y = \delta(t-1) - \delta(t-2) \quad \text{with } y(0) = y'(0) = 0.$$

Solution Taking the Laplace transform in the usual way and using result (15) gives

$$(s^2 + 3s + 2)Y(s) = e^{-s} - e^{-2s},$$

and so

$$Y(s) = \frac{e^{-s} - e^{-2s}}{s^2 + 3s + 2} = \frac{e^{-s} - e^{-2s}}{s+1} - \frac{e^{-s} - e^{-2s}}{s+2}.$$

Inverting the transform using Theorem 7.6 (the t-shift theorem) shows that

$$y(t) = H(t-1)[e^{1-t} - e^{2-2t}] - H(t-2)[e^{2-t} - e^{4-2t}].$$

A graph of this solution is given in Fig. 7.16. The graph shows that a physical system represented by the given differential equation subject to the equilibrium initial conditions $y(0) = y'(0) = 0$ is at rest until it is excited by the delta function at time $t = 1$ and then, after peaking just before $t = 2$, it is excited in the opposite sense by the delta function at time $t = 2$, after which the solution decays to zero as t increases, corresponding to the system returning to rest.

The Laplace transform is also discussed in references [3.4], [3.8], [3.9], [3.17], and [3.20]; tables of Laplace transform pairs are to be found in references [G.1], [G.3], [3.11], and [3.14]. An advanced account of the Laplace transform is to be found in reference [3.19]. ■

PAUL ADRIEN DIRAC (1902–1984)
An English mathematical physicist who introduced the delta function in a fundamental paper on quantum mechanics presented to the Royal Society of London in 1927. Together with the German physicist **Erwin Schrodinger** he shared the Nobel Prize for physics because of contributions made to quantum mechanics.

Summary

This section has been concerned with what are known as the operational properties of the Laplace transform. These are general properties of the transform itself that can be applied to any function $f(t)$ that possesses a Laplace transform, or to any function $F(s)$ that is the Laplace transform of a function $f(t)$. It will be seen later that these properties can be used to extend the table of Laplace transforms given at the end of Section 7.1, and when using the Laplace transform to solve differential equations.

EXERCISES 7.2

Exercises involving the transformation of derivatives

1. Prove that $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$.

2. Prove that $\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$.

3. Given that $f(0) = 1$, $f'(0) = 0$, $f''(0) = 1$, find $\mathcal{L}\{f'''(t)\}$.

4. Given that $f(0) = 0$, $f'(0) = 2$, $f''(0) = 2$, $f'''(0) = -4$, find $\mathcal{L}\{f^{(4)}(t)\}$.

5. Given that $f(t) = \begin{cases} \sin t, & 0 \le t < \pi/2 \\ t = 0, & t \ge \pi/2 \end{cases}$, find $\mathcal{L}\{f(t)\}$.

6. Given that $f(t) = \begin{cases} \sin t, & 0 \le t < \pi/2 \\ 1, & t \ge \pi/2 \end{cases}$, find $\mathcal{L}\{f(t)\}$.

7. Solve $y'' - 3y' + 2y = \cos t$, with $y(0) = 1$, $y'(0) = -1$.

8. Solve $y'' + 5y' + 4y = \exp(-t)$, with $y(0) = 1$, $y'(0) = 0$.

9. Solve $y'' + 8y' - 9y = t$, with $y(0) = 2$, $y'(0) = 1$.

10. Solve $y'' + 5y' + 6y = 1 + t^2$, with $y(0) = 0$, $y'(0) = 0$.

Exercises involving the first shift theorem (s-shift)

11. Find $\mathcal{L}\{(2 + t^3)e^{-2t}\}$.

12. Find $\mathcal{L}\{e^{-3t}\cos 2t\}$.

13. Find $\mathcal{L}\{e^{-t}t\sin 2t\}$.

14. Find $\mathcal{L}\{(1 + t^2)e^{-4t}\}$.

15. Find $\mathcal{L}\{e^{2t}\sin 3t\}$.

16. Find $\mathcal{L}\{e^{-4t}\sinh 3t\}$.

17. Find $\mathcal{L}^{-1}\{1/(s^2 - 4s + 13)\}$.

18. Find $\mathcal{L}^{-1}\{s/(s^2 + 4s + 13)\}$.

19. Find $\mathcal{L}^{-1}\{(1 - 3s)/(s^2 + 2s + 5)\}$.

20. Find $\mathcal{L}^{-1}\{1/[s(s^2 - 2s + 5)]\}$.

21. Find $\mathcal{L}^{-1}\{s/[(s + 1)(s^2 - 4s + 13)]\}$.

22. Find $\mathcal{L}^{-1}\{3/(s^2 + 6s + 25)\}$.

23. Find $\mathcal{L}^{-1}\{3(s^2 + 4)/[s(s^2 + 4s + 8)]\}$.

24. Find $\mathcal{L}^{-1}\{2/[(s + 3)^2(s^2 + 8s + 20)]\}$.

Exercises involving graphing functions with a t-shift

25. Sketch $f(t) = H(t - 2)(1 + t)$.

26. Sketch $f(t) = H(t - \pi)\sin t + H(t - 2\pi)$.

27. Sketch $f(t) = [H(t - \pi) - H(t - 2\pi)]\cos t$.

28. Sketch $f(t) = \sum_{r=0}^{4} H(t - r)$.

29. Sketch $f(t) = H(t - \pi)\cos(t - \pi)$.

30. Sketch $f(t) = H(t - 1)(t - 1)^2$.

31. Sketch $f(t) = [H(t - 1) - H(t - 2)](t - 1)^2$.

32. Sketch $f(t) = H(t - \pi/2)\cos(t - \pi/2)$.

Exercises involving the second shift theorem (t-shift)

33. Find $\mathcal{L}\{H(t - 3)(t - 3)^3\}$.

34. Find $\mathcal{L}\{H(t - 1)\sin(t - 1)\}$.

35. Find $\mathcal{L}\{H(t - 3\pi/2)\sin 2(t - 3\pi/2)\}$.

36. Find $\mathcal{L}\{H(t - \pi/2)(t - \pi/2)^3 - H(t - 3\pi/2) \times (t - 3\pi/2)^3\}$.

37. Find $\mathcal{L}\{H(t - 4)\sinh 3(t - 4)\}$.

38. Find $\mathcal{L}\{H(t - 1)(t - 1)\sin(t - 1)\}$.

39. Find $\mathcal{L}^{-1}\{s\,e^{-2s}/(s^2 + 4)\}$.

40. Find $\mathcal{L}^{-1}\{e^{-\pi s/3}/(s^2 + 9)\}$.

41. Find $\mathcal{L}^{-1}\{e^{-\pi s/2}(s + 1)/(s^2 + 4s + 5)\}$.

42. Find $\mathcal{L}^{-1}\{e^{-2s}(s^2 + s + 1)/[s(s + 2)^2]\}$.

43. Find $\mathcal{L}^{-1}\{e^{-4s}(s + 3)/(s^2 + 4s + 13)\}$.

44. Find $\mathcal{L}^{-1}\{e^{-3s}s^2/[s(s^2 + 4s + 8)]\}$.

45. Solve $y'' + 5y' + 6y = H(t - \pi)\cos(t - \pi)$, with $y(0) = 1$, $y'(0) = 0$.

46. Solve $y'' - 5y' + 6y = tH(t - 1)$, with $y(0) = 0$, $y'(0) = 0$.

47. Solve $y'' - 5y' + 6y = 1 + tH(t - 2)$, with $y(0) = 0$, $y'(0) = 1$.

48. Solve $y'' - 6y' + 10y = tH(t - 3)$, with $y(0) = 1$, $y'(0) = 1$.

49. Solve $y'' + 2y' + 10y = e^{-t}H(t - 1)$, with $y(0) = -1$, $y'(0) = 0$.

50. Solve $y'' - y' - 2y = e^{-t}H(t - 1)$, with $y(0) = 1$, $y'(0) = 0$.

Exercises involving differentiation of transforms

51. Find $\mathcal{L}\{t^2e^{3t}\sin t\}$.

52. Find $\mathcal{L}\{te^{-t}\sin 4t\}$.

53. Find $\mathcal{L}\{t^3e^{2t}\sin 2t\}$.

54. Find $\mathcal{L}\{t^2e^{3t}\cos 2t\}$.

Exercises involving scaling

55. If $\mathcal{L}\{f(t)\} = e^{-3s}(s^2 - 1)/(s^4 - a^4)$, find $\mathcal{L}\{f(2t)\}$.

56. If $\mathcal{L}\{f(t)\} = (s + 1)(s^2 + 2)/(s^2 + 4)^2$, find $\mathcal{L}\{f(3t)\}$.

57. If $\mathcal{L}\{f(t)\} = 1/[s^2(s^2 + 4)]$, find $\mathcal{L}\{f(t/3)\}$.

58. If $\mathcal{L}\{f(t)\} = (s^2 - 4)/[(s^2 + 4)^2]$, find $\mathcal{L}\{f(t/2)\}$.

Exercises involving the Laplace transform of periodic functions

In Exercises 59 through 66 find the Laplace transform of the periodic function $f(t)$.

59.

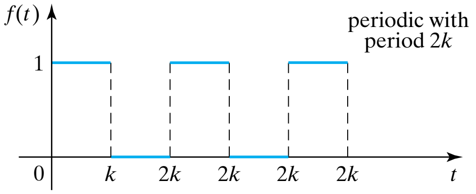


FIGURE 7.17

60.

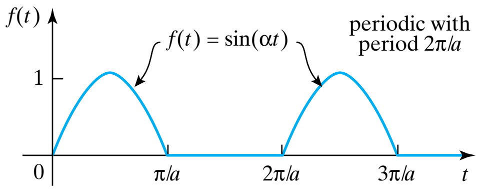


FIGURE 7.18

61.

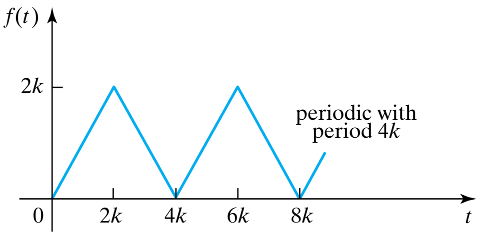


FIGURE 7.19

62.

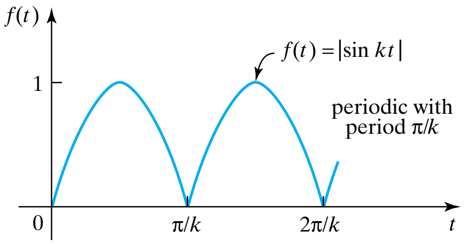


FIGURE 7.20

63.

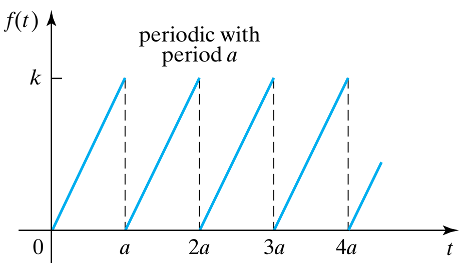


FIGURE 7.21

64.

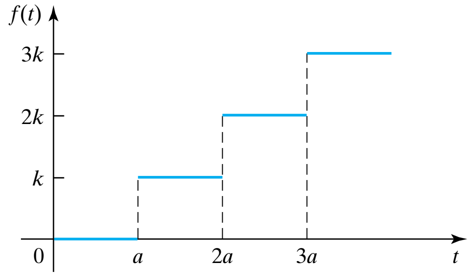


FIGURE 7.22

65.

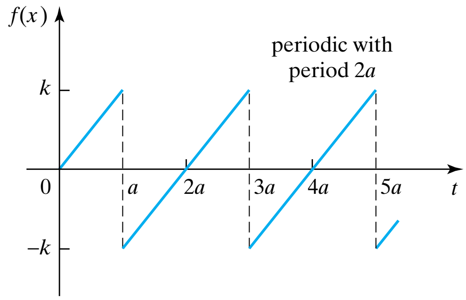


FIGURE 7.23

66.

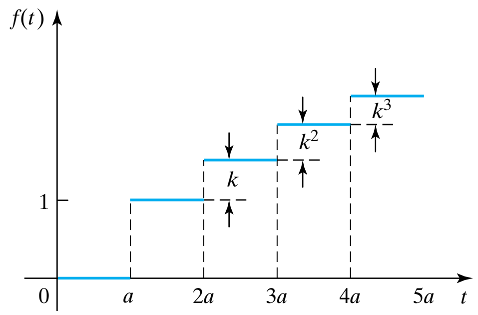


FIGURE 7.24

Exercises involving the convolution operation

67. Find $(e^{-t} * e^{-2t})$.

68. Find $(t * \sin t)$.

69. Find $(t^2 * \sin t)$.

70. Find $(t * e^{-t})$.

71. Find $(\cos t * \cos t)$.

72. Find $(\sin 2t * \sin 2t)$.

Exercises involving the convolution theorem

73. Find $\mathcal{L}\{t * e^{-2t}\}$.

74. Find $\mathcal{L}\{2t * \cos 2t\}$.

75. Find $\mathcal{L}\{e^{-t} \sin t * t\}$.

76. Find $\mathcal{L}\{e^{-2t} \cos t * e^{t}\}$.

77. Find $\mathcal{L}^{-1}\{1/[s^2(s^2+4)]\}$.

78. Find $\mathcal{L}^{-1}\{1/(s^2-9)^2\}$.

79. Find $\mathcal{L}^{-1}\{s^2/(s^2-1)^2\}$.

80. Find $\mathcal{L}^{-1}\{s/(s^2-4)^2\}$.

Exercises involving integral equations

81. Solve $y(t) = \sin t + \int_0^t \sin(t-\tau)y(\tau)d\tau$.

82. Solve $y(t) = \cos t + \int_0^t \sin[2(t-\tau)]y(\tau)d\tau$.

83. Solve $y(t) = t^2 + \int_0^t \cos(t-\tau)y(\tau)d\tau$.

84. Solve $y(t) = e^{-2t} + \int_0^t \cos(t-\tau)y(\tau)d\tau$.

Exercises involving integro-differential equations

85. Solve $y' + 4y = 4\int_0^t \sin\tau\, y(t-\tau)d\tau$, with $y(0) = 1$.

86. Solve $y' + y = \int_0^t e^{-2\tau} y(t-\tau)d\tau$, with $y(0) = 3$.

87. Solve $y'' - y = \int_0^t \sinh\tau\, y(t-\tau)d\tau$, with $y(0) = 1$, $y'(0) = 0$.

88. Solve $y'' - 4y = 2\int_0^t \sinh 2\tau\, y(t-\tau)d\tau$, with $y(0) = 1$, $y'(0) = 0$.

Exercises involving the transform of an integral

89. Find $\mathcal{L}\left\{\int_0^t \tau^2 \sin 2\tau\, d\tau\right\}$.

90. Find $\mathcal{L}\left\{\int_0^t e^{2\tau} \cos\tau\, d\tau\right\}$.

91. Find $\mathcal{L}^{-1}\{1/(s^2+a^2)^2\}$.

92. Find $\mathcal{L}^{-1}\{s/(s^2+a^2)\}$.

Exercises involving an integral of a transform

93. Find $\mathcal{L}\left\{\dfrac{\sinh 2t}{t}\right\}$.

94. Find $\mathcal{L}\left\{\dfrac{1-\cos 3t}{t}\right\}$.

95. Find $\mathcal{L}^{-1}\left\{\ln\left(\dfrac{s^2-a^2}{s^2}\right)\right\}$.

96. Find $\mathcal{L}^{-1}\left\{\ln\left(\dfrac{s^2+a^2}{s^2}\right)\right\}$.

Exercises involving the initial value theorem

In Exercises 97 through 100 use the initial value theorem to find $f(0)$, $f'(0)$, and $f''(0)$ from $F(s)$, and verify the result by differentiation of $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

97. $F(s) = (s^2+6)/\{s(s^2+9)\}$.

98. $F(s) = s/(s^2+6s+9)$.

99. $F(s) = (s-1)/(s^2-4s+4)$.

100. $F(s) = (2s^2+s-12)/\{s(s+2)(s+3)\}$.

Exercises involving the delta function

101. Evaluate $\int_0^\infty \left(\dfrac{1-3\sin^2 t}{t}\right)\delta(t-\pi/2)dt$.

102. Evaluate $\int_0^4 \sin^2 t\,\delta(t-2\pi)dt$.

103. Evaluate $\int_0^\infty \sum_{n=1}^{3}\left\{\left(\dfrac{\sin nt}{t}\right)\delta\left[t-(2n+1)\dfrac{\pi}{2}\right]\right\}dt$.

104. Evaluate $\int_0^\infty \{[H(t-1)-H(t-2)]t + \cos(t-3\pi)\delta(t-3\pi)\}dt$.

105. Solve $y'' + 9y = 1 + \delta(t-1)$, with $y(0)=0$, $y'(0)=0$.

106. Solve $y'' + 4y' + 4y = \delta(t-1)$, with $y(0)=1$, $y'(0) = 1$.

107. Solve $y'' + 2y' + y = \sin t + \delta(t-\pi)$, with $y(0) = y'(0) = 0$.

108. Solve $y'' - 4y' + 3y = e^{-t} + 3\delta(t-2)$, with $y(0) = y'(0) = 0$.

109. Solve $y'' + 4y = 1 - H(t-1) + \delta(t-2)$, with $y(0) = 1$, $y'(0) = 0$.

110. Solve $y'' + 3y' + 2y = \delta(t-1)$, with $y(0) = 0$, $y'(0) = 1$.

7.3 Systems of Equations and Applications of the Laplace Transform

(a) Solution of Systems of Linear First Order Equations by the Laplace Transform

The Laplace transform can be used to solve initial value problems for systems of linear first order differential equations by introducing the Laplace transform of

each dependent variable that is involved, solving the resulting algebraic equations for each transformed dependent variable, and then inverting the results.

As a system of linear higher order differential equations can always be reduced to a system of first order equations by introducing higher order derivatives as new dependent variables, the solution of a system of linear first order equations can be considered to be the most general case.

solving systems of equations

The example that follows, involving two simultaneous first order equations, illustrates the approach to be used in all cases, but by restricting the number of equations and using simple nonhomogeneous terms (forcing functions) the algebra is kept to a minimum.

EXAMPLE 7.29

Solve the initial value problem

$$x' - 2x + y = \sin t$$
$$y' + 2x - y = 1,$$

with $x(0) = 1,\ y(0) = -1$.

Solution We define the transforms of the dependent variables $x(t)$ and $y(t)$ to be

$$\mathcal{L}\{x(t)\} = X(s), \quad \mathcal{L}\{y(t)\} = Y(s).$$

Transforming the system of equations in the usual way leads to the following system of linear algebraic equations for $X(s)$ and $Y(s)$:

$$sX(s) - 1 - 2X(s) + Y(s) = 1/(s^2+1)$$
$$sY(s) + 1 + 2X(s) - Y(s) = 1/s.$$

Solving these for $X(s)$ and $Y(s)$ gives

$$X(s) = \frac{(s-1)(s^3+s^2+2s+1)}{s^2(s-3)(s^2+1)} \quad \text{and} \quad Y(s) = \frac{-(s^4-s^3+3s^2+s+2)}{s^2(s-3)(s^2+1)}.$$

Expressing these results in terms of partial fractions, we find that

$$X(s) = \frac{4}{9}\frac{1}{s} + \frac{1}{3}\frac{1}{s^2} - \frac{1}{5}\frac{1}{s^2+1} - \frac{2}{5}\frac{s}{s^2+1} + \frac{43}{45}\frac{1}{s-3}$$

and

$$Y(s) = \frac{5}{9}\frac{1}{s} + \frac{2}{3}\frac{1}{s^2} + \frac{1}{5}\frac{1}{s^2+1} - \frac{3}{5}\frac{s}{s^2+1} - \frac{43}{45}\frac{1}{s-3}.$$

Finally, taking the inverse transform gives the solution

$$x(t) = \frac{4}{9} + \frac{1}{3}t - \frac{1}{5}\sin t - \frac{2}{5}\cos t + \frac{43}{45}e^{3t}$$

and

$$y(t) = \frac{5}{9} + \frac{2}{3}t + \frac{1}{5}\sin t - \frac{3}{5}\cos t - \frac{43}{45}e^{3t} \quad \text{for } t > 0.$$ ■

This method can be used for any number of simultaneous linear differential equations, though the complexity of both the algebraic manipulation and the associated inversion problem increases rapidly when more than two equations are involved.

A typical example of the way systems of first order equations arise in practice is provided by considering a chemical reaction that converts a raw chemical into an end product, via several intermediate reactions. The simplest situation involves chemical reactions that are irreversible, so that once a product has been produced the chemical process cannot be reversed, causing the new product to revert to a previous one.

Let us derive the system of equations governing such a process when three intermediate reactions are involved, each of which is irreversible, with each reaction proceeding at a rate that is proportional to the amount of material to be converted from one stage to the next. Denote the raw chemical by A and the end product by E, with the intermediate products denoted by B, C, and D, and let the reaction rates (the constants of proportionality) from $A \to B$, $B \to C$, $C \to D$, and $D \to E$ be k_1, k_2, k_3, and k_4, respectively. Then if the amounts of chemicals A, B, C, D, and E present at time t are x, y, u, v, and w, the production and removal of the chemical products involved is described as follows.

Reaction	Reaction Rate of Removal	Reaction Rate of Production
$A \to B$	$\left(\frac{dx}{dt}\right)_{A\to B} = -k_1 x$	$\left(\frac{dy}{dt}\right)_{A\to B} = k_1 x$
$B \to C$	$\left(\frac{dy}{dt}\right)_{B\to C} = -k_2 y$	$\left(\frac{du}{dt}\right)_{B\to C} = k_2 y$
$C \to D$	$\left(\frac{du}{dt}\right)_{C\to D} = -k_3 u$	$\left(\frac{dv}{dt}\right)_{C\to D} = k_3 u$
$D \to E$	$\left(\frac{dv}{dt}\right)_{D\to E} = -k_4 v$	$\left(\frac{dw}{dt}\right)_{D\to E} = k_4 v$

Combining these results gives

$$\begin{aligned}
\frac{dx}{dt} &= \left(\frac{dx}{dt}\right)_{A\to B} = -k_1 x \\
\frac{dy}{dt} &= \left(\frac{dy}{dt}\right)_{A\to B} + \left(\frac{dy}{dt}\right)_{B\to C} = k_1 x - k_2 y \\
\frac{du}{dt} &= \left(\frac{du}{dt}\right)_{B\to C} + \left(\frac{du}{dt}\right)_{C\to D} = k_2 y - k_3 u \\
\frac{dv}{dt} &= \left(\frac{dv}{dt}\right)_{C\to D} + \left(\frac{dv}{dt}\right)_{D\to E} = k_3 u - k_4 v.
\end{aligned}$$

If the amount of raw material A present at the start is Q, the initial conditions for the system are seen to be

$$x(0) = Q, \quad y(0) = 0, \quad u(0) = 0, \quad v(0) = 0, \quad \text{and} \quad w(0) = 0.$$

Provided no additional by-products are produced during the reactions, it follows from the conservation of mass that $x + y + u + v + w = Q$, and so

$$w = Q - x - y - u - v.$$

Taking the Laplace transform of this system of first order linear equations and using the stated initial conditions leads to the transformed system

$$\begin{aligned} sX(s) + k_1 X(s) &= Q \\ sY(s) - k_1 X(s) + k_2 Y(s) &= 0 \\ sU(s) - k_2 Y(s) + k_3 U(s) &= 0 \\ sV(s) - k_3 U(s) + k_4 V(s) &= 0, \end{aligned}$$

where $\mathcal{L}\{x(t)\} = X(s)$, $\mathcal{L}\{y(t)\} = Y(s)$, $\mathcal{L}\{u(t)\} = U(s)$, and $\mathcal{L}\{v(t)\} = V(s)$.

Solving for the Laplace transforms, we have

$$X(s) = \frac{Q}{s+k_1}, \quad Y(s) = \frac{k_1 Q}{(s+k_1)(s+k_2)}, \quad U(s) = \frac{k_1 k_2 Q}{(s+k_1)(s+k_2)(s+k_3)},$$

and

$$V(s) = \frac{k_1 k_2 k_3 Q}{(s+k_1)(s+k_2)(s+k_3)(s+k_4)}.$$

After expressing these Laplace transforms in terms of partial fractions the required solutions are seen to be

$$x(t) = Qe^{-k_1 t}, \quad y(t) = \frac{k_1 Q}{k_1 - k_2}(e^{-k_1 t} - e^{-k_2 t})$$

and

$$u(t) = k_1 k_2 Q \left(\frac{1}{(k_2 - k_1)(k_3 - k_1)} e^{-k_1 t} + \frac{1}{(k_1 - k_2)(k_3 - k_2)} e^{-k_2 t} + \frac{1}{(k_1 - k_3)(k_2 - k_3)} e^{-k_3 t} \right)$$

with $v(t)$ similarly defined. The amount of the end product $w(t)$ produced at time t follows from

$$w(t) = Q - x(t) - y(t) - u(t) - v(t).$$

solving systems of equations in matrix form

We now outline a matrix method of solution of initial value problems for systems of linear first order differential equations, of which Example 7.29 is a typical case. Let us consider the system

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}(t), \tag{17}$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ & \cdots & \cdots & \cdots & & \\ & \cdots & \cdots & \cdots & & \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nn} \end{bmatrix}, \quad \mathbf{b}(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \cdot \\ \cdot \\ b_n(t) \end{bmatrix},$$

subject to the initial conditions $x_1(0) = x_1, x_2(0) = x_2, \ldots, x_n(0) = x_n$.

Define $\mathcal{L}\{x_1(t)\} = X_1(s)$, $\mathcal{L}\{x_2(t)\} = X_2(s)\ldots$, $\mathcal{L}\{x_n(t)\} = X_n(s)$, $\mathcal{L}\{b_1(t)\} = B_1(s)$, $\mathcal{L}\{b_2(t)\} = B_2(s), \ldots, \mathcal{L}\{b_n(t)\} = B_n(s)$, and set

$$\mathbf{Z}(s) = \begin{bmatrix} X_1(s) \\ X_2(s) \\ \cdot \\ \cdot \\ X_n(s) \end{bmatrix}, \quad \mathbf{c}(s) = \begin{bmatrix} B_1(s) \\ B_2(s) \\ \cdot \\ \cdot \\ B_n(s) \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}.$$

Then taking the Laplace transform of (17) and using the result $\mathcal{L}\{x_r'(t)\} = s\,X(s) - x_r$, for $r = 1,\ 2, \ldots, n$, we arrive at the system

$$s\mathbf{Z}(s) - \mathbf{v} = \mathbf{AZ}(s) + \mathbf{c}(s)$$

or, equivalently,

$$(s\mathbf{I} - \mathbf{A})\mathbf{Z}(s) = \mathbf{v} + \mathbf{c}(s),$$

where $\mathbf{I}$ is the $n \times n$ unit matrix. Premultiplying this last result by $(s\mathbf{I} - \mathbf{A})^{-1}$ gives

$$\mathbf{Z}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}[\mathbf{v} + \mathbf{c}(s)]. \tag{18}$$

Finally, taking the inverse Laplace transform of (18) we obtain the solution $\mathbf{x}(t)$ of the initial value problem in the form

$$\mathbf{x}(t) = \mathcal{L}^{-1}\{[s\mathbf{I} - \mathbf{A}]^{-1}[\mathbf{v} + \mathbf{c}(s)]\}. \tag{19}$$

EXAMPLE 7.30 Solve the initial value problem of Example 7.29 by using result (19).

Solution Making the necessary identifications we have

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{c}(s) = \begin{bmatrix} 1/(s^2+1) \\ 1/s \end{bmatrix},$$

so (18) becomes

$$\mathbf{Z}(s) = \left[s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}\right]^{-1}\left[\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1/(s^2+1) \\ 1/s \end{bmatrix}\right],$$

or

$$\mathbf{Z}(s) = \begin{bmatrix} s-2 & 1 \\ 2 & s-1 \end{bmatrix}^{-1}\begin{bmatrix} (s^2+2)/(s^2+1) \\ (1-s)/s \end{bmatrix}.$$

The inverse of the first matrix in this product is

$$\begin{bmatrix} s-2 & 1 \\ 2 & s-1 \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{s-1}{s(s-3)} & \dfrac{-1}{s(s-3)} \\ \dfrac{-2}{s(s-3)} & \dfrac{s-2}{s(s-3)} \end{bmatrix},$$

so

$$\mathbf{Z}(s) = \begin{bmatrix} \dfrac{s-1}{s(s-3)} & \dfrac{-1}{s(s-3)} \\ \dfrac{-2}{s(s-3)} & \dfrac{s-2}{s(s-3)} \end{bmatrix}\begin{bmatrix} \dfrac{s^2+2}{s^2+1} \\ \dfrac{1-s}{s} \end{bmatrix}.$$

After forming the matrix product this becomes

$$\mathbf{Z}(s) = \begin{bmatrix} \dfrac{(s-1)(s^3+s^2+2s+1)}{s^2(s-3)(s^2+1)} \\ \dfrac{-(s^4-s^3+3s^2+s+2)}{s^2(s-3)(s^2+1)} \end{bmatrix}.$$

The inverse transforms involved are, of course, the same as the ones in Example 7.29, so, as would be expected, the solution is the same as before, apart from a change of notation involving the replacement of $x(t)$ and $y(t)$ by $x_1(t)$ and $x_2(t)$ giving

$$x_1(t) = \frac{4}{9} + \frac{1}{3}t - \frac{1}{5}\sin t - \frac{2}{5}\cos t + \frac{43}{45}e^{3t}$$

and

$$x_2(t) = \frac{5}{9} + \frac{2}{3}t + \frac{1}{5}\sin t - \frac{3}{5}\cos t - \frac{43}{45}e^{3t} \quad \text{for } t > 0.$$ ■

(b) Determination of $e^{t\mathbf{A}}$ by Means of the Laplace Transform

The matrix solution of system (17) given in (19) has an interesting and useful consequence, because it provides a different and efficient way of finding the matrix exponential $e^{t\mathbf{A}}$. To see how this comes about, notice that from equation (114) in Section 6.10(c) the solution of the homogeneous system of equations

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{20}$$

subject to the initial condition $\mathbf{x}(0) = \mathbf{v}$, can be written

$$\mathbf{x}(t) = e^{t\mathbf{A}}\,\mathbf{v}. \tag{21}$$

Setting $\mathbf{c}(s) = \mathbf{0}$ (corresponding to $\mathbf{b}(t) = \mathbf{0}$) reduces solution (19) to

$$\mathbf{x}(t) = \mathcal{L}^{-1}\{[s\mathbf{I} - \mathbf{A}]^{-1}\}\mathbf{v}, \tag{22}$$

so comparison of (21) and (22) shows that

$$e^{t\mathbf{A}} = \mathcal{L}^{-1}\{[s\mathbf{I} - \mathbf{A}]^{-1}\}. \tag{23}$$

We have established the following theorem.

THEOREM 7.15

finding the matrix exponential by the Laplace transform

Determination of $e^{t\mathbf{A}}$ by means of the Laplace transform Let $\mathbf{A}$ be a real $n \times n$ matrix with constant elements. Then the exponential matrix

$$e^{t\mathbf{A}} = \mathcal{L}^{-1}\{[s\mathbf{I} - \mathbf{A}]^{-1}\}.$$ ■

The following examples show how Theorem 7.15 determines $e^{t\mathbf{A}}$ in the cases when $\mathbf{A}$ is diagonalizable with real eigenvalues, when it is diagonalizable with complex conjugate eigenvalues, and also when it is not diagonalizable.

EXAMPLE 7.31

Use Theorem 7.15 to find $e^{t\mathbf{A}}$ when

$$\mathbf{A} = \begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix}.$$

Solution Matrix $\mathbf{A}$ has the distinct eigenvalues 1 and 2, and so is diagonalizable.

$$[s\mathbf{I}-\mathbf{A}] = \begin{bmatrix} s+2 & -6 \\ 2 & s-5 \end{bmatrix}$$

so

$$[s\mathbf{I}-\mathbf{A}]^{-1} = \begin{bmatrix} \dfrac{s-5}{s^2-3s+2} & \dfrac{6}{s^2-3s+2} \\ \dfrac{-2}{s^2-3s+2} & \dfrac{s+2}{s^2-3s+2} \end{bmatrix}.$$

Expressing each element of this matrix in terms of partial fractions and taking the inverse Laplace transform gives

$$e^{t\mathrm{A}} = \begin{bmatrix} 4e^t - 3e^{2t} & -6e^t + 6e^{2t} \\ 2e^t - 2e^{2t} & -3e^t + 4e^{2t} \end{bmatrix},$$

in agreement with the result in Example 6.33. ■

EXAMPLE 7.32 Use Theorem 7.14 to find $e^{t\mathrm{A}}$ when

$$\mathbf{A} = \begin{bmatrix} -3 & -4 \\ 2 & 1 \end{bmatrix}.$$

Solution Matrix $\mathbf{A}$ has the complex conjugate eigenvalues $-1 \pm 2i$.

$$[s\mathbf{I}-\mathbf{A}] = \begin{bmatrix} s+3 & 4 \\ -2 & s-1 \end{bmatrix},$$

so

$$[s\mathbf{I}-\mathbf{A}]^{-1} = \begin{bmatrix} \dfrac{s-1}{s^2+2s+5} & \dfrac{-4}{s^2+2s+5} \\ \dfrac{2}{s^2+2s+5} & \dfrac{s+3}{s^2+2s+5} \end{bmatrix}.$$

Expressing each element of this matrix in terms of partial fractions and taking the inverse Laplace transform gives

$$e^{t\mathrm{A}} = \begin{bmatrix} e^{-t}(\cos 2t - \sin 2t) & -2e^{-t}\sin 2t \\ e^{-t}\sin 2t & e^{-t}(\cos 2t + \sin 2t) \end{bmatrix},$$

in agreement with the result of Example 6.34. ■

EXAMPLE 7.33 Use Theorem 7.14 to find $e^{t\mathrm{A}}$ when

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}.$$

Solution Matrix $\mathbf{A}$ has the repeated eigenvalue 4 and is not diagonalizable.

$$[s\mathbf{I}-\mathbf{A}] = \begin{bmatrix} s-4 & -1 \\ 0 & s-4 \end{bmatrix},$$

so

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \begin{bmatrix} \dfrac{1}{s-4} & \dfrac{1}{(s-4)^2} \\ 0 & \dfrac{1}{s-4} \end{bmatrix}.$$

Taking the inverse of the elements of this matrix, we find that

$$e^{t\mathrm{A}} = \begin{bmatrix} e^{4t} & te^{4t} \\ 0 & e^{4t} \end{bmatrix},$$

in agreement with the result of Example 6.35. ■

(c) The Weighting Function

To introduce the concept of a *weighting function*, which has important engineering applications, we consider the differential equation

$$a_0 \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y = f(t), \tag{24}$$

weighting function and its uses

subject to the initial conditions $y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0$. We shall denote by $w(t)$ the solution of equation (24) when $f(t) = \delta(t)$, and call it the **weighting function** associated with the equation. Thus the solution $w(t)$ can be regarded as the *output* from a system described by equation (24) that is produced by the impulsive *input* (nonhomogeneous term) $\delta(t)$ applied at time $t = 0$ when the system is at rest. The weighting function $w(t)$ is the solution of the equation

$$a_0 \frac{d^n w}{dt^n} + a_1 \frac{d^{n-1} w}{dt^{n-1}} + \cdots + a_n w = \delta(t), \tag{25}$$

with $w(t) = 0$ for $t < 0$.

Let us now consider the *output* $y(t)$ from a system described by (24) produced by an arbitrary *input* $f(t)$, subject to the homogeneous initial conditions $y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0$. Taking the Laplace transform of (24) we find that

$$G(s)Y(s) = F(s), \tag{26}$$

where

$$G(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n, \quad Y(s) = \mathcal{L}\{y(t)\} \quad \text{and} \quad F(s) = \mathcal{L}\{f(t)\}.$$

Setting $W(s) = \mathcal{L}\{w(t)\}$, taking the Laplace transform of (25), and using the fact that $w(t)$ and all its derivatives vanish for $t < 0$ leads to the result

$$G(s)W(s) = 1. \tag{27}$$

Eliminating $G(s)$ between (26) and (27) relates the Laplace transform of the output $Y(s)$ to the Laplace transform $F(s)$ of the input by the equation

$$Y(s) = W(s)F(s). \tag{28}$$

Taking the inverse Laplace transform of (28) and using the convolution theorem gives

$$y(t) = \int_0^t w(\tau) f(t-\tau) d\tau. \tag{29}$$

This form of the solution of (24) explains why $w(t)$ is called the *weighting function*, because (29) shows how the input $y(t-\tau)$ at time $t-\tau$ is *weighted* by the function $w(\tau)$ over the interval $0 \le \tau \le t$ in the integral determining $y(t)$.

The determination of the weighting function has the advantage that once it has been found, the solution of (24), subject to the conditions that $y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0$, is always expressible as result (29) for every nonhomogeneous term $f(t)$. It is instructive to compare this result, which applies to a linear differential equation of *any* order, to the one in (76) of Section 6.6, which was obtained by applying the method of variation of parameters to a second order equation with homogeneous initial conditions when $t = a$. The weighting function is also sometimes called the *Green's function* for an initial value problem for a homogeneous differential equation.

The modification that must be made to result (29) to take account of initial conditions for $y(t)$ that are not all zero at $t = 0$ is to be found in Exercise 25 at the end of this section.

EXAMPLE 7.34 Find the weighting function for the equation

$$y'' + 2y' + 5y = \sin t$$

and use it to solve the equation subject to the initial condition $y(0) = y'(0) = 0$.

Solution The weighting function $w(t)$ is the solution of

$$w'' + 2w' + 5w = \delta(t)$$

with $w(0) = w'(0) = 0$. Taking the Laplace transform and setting $\mathcal{L}\{w(t)\} = W(s)$ gives

$$s^2 W(s) + 2s W(s) + 5W(s) = 1,$$

so

$$W(s) = \frac{1}{s^2 + 2s + 5}.$$

Taking the inverse Laplace transform, we find that

$$w(t) = \mathcal{L}^{-1}\{W(s)\} = \frac{1}{2} e^{-t} \sin 2t \quad \text{for } t \ge 0.$$

The solution of the differential equation with $y(0) = y'(0) = 0$ now follows from (29) as

$$\begin{aligned} y(t) &= \int_0^t w(\tau) \sin(t-\tau) d\tau \\ &= \frac{1}{2} \int_0^t e^{-\tau} \sin 2\tau \sin(t-\tau) d\tau \\ &= \frac{1}{5} \sin t - \frac{1}{10} \cos t + \frac{e^{-t}}{20}(2\cos 2t - \sin 2t). \end{aligned}$$

■

The concept of a weighting function can be generalized to include systems of equations, though then more than one weighting function must be introduced, and the solution of each dependent variable becomes the sum of convolution integrals of the type given in (29). The ideas involved are illustrated by considering the following system of equations involving $x(t)$ and $y(t)$:

$$\begin{aligned} x' + ax + by &= f_1(t) \\ y' + cx + dy &= f_2(t), \end{aligned} \tag{30}$$

subject to the initial conditions $x(0) = y(0) = 0$.

It is necessary to introduce a weighting function for each of the variables $x(t)$ and $y(t)$ corresponding first to $f_1(t) = \delta(t)$ and $f_2(t) = 0$, and then to $f_1(t) = 0$ and $f_2(t) = \delta(t)$. Let $w_{x1}(t)$ and $w_{y1}(t)$ be the weighting functions corresponding to

$$\begin{aligned} w'_{x1} + aw_{x1} + bw_{y1} &= \delta(t) \\ w'_{y1} + cw_{x1} + dw_{y1} &= 0, \end{aligned} \tag{31}$$

and $w_{x2}(t)$ and $w_{y2}(t)$ be the Green's functions corresponding to

$$\begin{aligned} w'_{x2} + aw_{x2} + bw_{y2} &= 0 \\ w'_{y2} + cw_{x2} + dw_{y2} &= \delta(t), \end{aligned} \tag{32}$$

where $w_{x1}(0) = w_{x2}(0) = w_{y1}(0) = w_{y2}(0) = 0$.

The notation used here indicates that $w_{x1}(t)$ is the x response and $w_{y1}(t)$ the y response to the input $f_1(t) = \delta(t)$ and $f_2(t) = 0$, and $w_{x2}(t)$ is the x response and $w_{y2}(t)$ the y response to the input $f_1(t) = 0$ and $f_2(t) = \delta(t)$. Then, because the equations are linear, to obtain the solution $x(t)$ subject to the initial conditions $x(0) = y(0) = 0$, it is necessary to add the contribution due to $w_{x1}(t)$ to the one due to $w_{x2}(t)$, and similarly for the solution $y(t)$.

This leads to the solution in the form

$$x(t) = \int_0^t w_{x1}(\tau) f_1(t-\tau)d\tau + \int_0^t w_{x2}(\tau) f_2(t-\tau)d\tau \tag{33a}$$

and

$$y(t) = \int_0^t w_{y1}(\tau) f_1(t-\tau)d\tau + \int_0^t w_{y2}(\tau) f_2(t-\tau)d\tau. \tag{33b}$$

Once the weighting functions have been found, equations (33) give the solution of system (30) for any choice of functions $f_1(t)$ and $f_2(t)$, subject to the initial conditions $x(0) = y(0) = 0$.

EXAMPLE 7.35 Find weighting functions for the equations

$$\begin{aligned} x' + 2x - y &= f_1(t) \\ y' - 2x + y &= f_2(t) \end{aligned}$$

and use them to solve the system subject to the initial conditions $x(0) = y(0) = 0$ when (a) $f_1(t) = \sin t$ and $f_2(t) = 2$ and (b) $f_1(t) = \cos t$ and $f_2(t) = 0$.

Solution **(a)** From (31) the functions $w_{x1}(t)$ and $w_{y1}(t)$ satisfy

$$\begin{aligned} w'_{x1} + 2w_{x1} - w_{y1} &= \delta(t) \\ w'_{y1} - 2w_{x1} + w_{y1} &= 0, \end{aligned}$$

so taking the Laplace transform of these equations we have

$$(s+2)\mathcal{L}\{w_{x1}(t)\} - \mathcal{L}\{w_{y1}(t)\} = 1$$
$$(s+1)\mathcal{L}\{w_{y1}(t)\} - 2\mathcal{L}\{w_{x1}(t)\} = 0.$$

Solving for $\mathcal{L}\{w_{x1}(t)\}$ and $\mathcal{L}\{w_{y1}(t)\}$ gives

$$\mathcal{L}\{w_{x1}(t)\} = \frac{s+1}{s(s+3)} \quad \text{and} \quad \mathcal{L}\{w_{y1}(t)\} = \frac{2}{s(s+3)}.$$

Taking the inverse Laplace transforms, we find that

$$w_{x1}(t) = \frac{1}{3} + \frac{2}{3}e^{-3t} \quad \text{and} \quad w_{y1}(t) = \frac{2}{3} - \frac{2}{3}e^{-3t} \quad \text{for } t \geq 0.$$

Similarly, solving the equations for $w_{x2}(t)$ and $w_{y2}(t)$ corresponding to (32), we obtain

$$w_{x2}(t) = \frac{1}{3} - \frac{1}{3}e^{-3t} \quad \text{and} \quad w_{y2}(t) = \frac{2}{3} + \frac{1}{3}e^{-3t} \quad \text{for } t \geq 0.$$

The solution of the system subject to the initial conditions $x(0) = y(0) = 0$, $f_1(t) = \sin t$, and $f_2(t) = 2$ now follows from (33) as

$$x(t) = \int_0^t w_{x1}(\tau)\sin(t-\tau)d\tau + 2\int_0^t w_{x2}(\tau)d\tau$$

and

$$y(t) = \int_0^t w_{y1}(\tau)\sin(t-\tau)d\tau + 2\int_0^t w_{y2}(\tau)d\tau.$$

After the integrations are performed, the solution is found to be

$$x(t) = \frac{1}{9} + \frac{2}{3}t + \frac{13}{45}e^{-3t} + \frac{1}{5}\sin t - \frac{2}{5}\cos t$$

and

$$y(t) = \frac{8}{9} + \frac{4}{3}t - \frac{13}{45}e^{-3t} - \frac{1}{5}\sin t - \frac{3}{5}\cos t \quad \text{for } t > 0.$$

(b) Similarly, the solution when $f_1(t) = \cos t$ and $f_2(t) = 0$ is given by

$$x(t) = \int_0^t w_{x1}(\tau)\cos(t-\tau)d\tau$$

and

$$y(t) = \int_0^t w_{y1}(\tau)\cos(t-\tau)d\tau,$$

so after performing the integrations,

$$x(t) = -\frac{1}{5}e^{-3t} + \frac{2}{5}\sin t + \frac{1}{5}\cos t$$

and

$$y(t) = \frac{1}{5}e^{-3t} + \frac{3}{5}\sin t - \frac{1}{5}\cos t \quad \text{for } t > 0.$$ ■

(d) Differential Equations with Polynomial Coefficients

special variable coefficient differential equations

The Laplace transform can be applied to linear differential equations with polynomial coefficients to find the solution of an initial value problem in the usual way, and also to deduce the Laplace transform of a function from its defining differential equation. This last situation is useful when the integral defining the Laplace transform of a function $f(t)$ cannot be evaluated directly. First, however, we use Theorems 7.3 and 7.7 to find the transform of a product of a power of t and a derivative of $f(t)$.

THEOREM 7.16

$\mathcal{L}\{t^m f^{(n)}(t)\}$ Let $f(t)$ be n times differentiable with $\mathcal{L}\{f(t)\} = F(s)$. Then

$$\mathcal{L}\{t^m f^{(n)}(t)\} = (-1)^m \frac{d^m}{ds^m}[s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{(n-1)}(0)].$$

Useful special cases are:

(i) $\mathcal{L}\{t f(t)\} = -F'(s)$

(ii) $\mathcal{L}\{t f'(t)\} = -sF'(s) - F(s)$

(iii) $\mathcal{L}\{t f''(t)\} = -s^2 F'(s) - 2sF(s) + f(0)$

(iv) $\mathcal{L}\{t^2 f'(t)\} = sF''(s) + 2F'(s)$

(v) $\mathcal{L}\{t^2 f''(t)\} = s^2 F''(s) + 4sF'(s) + 2F(s)$

Proof The results of the theorem are direct consequences of Theorems 7.3 and 7.7. We prove the general result, from which the special cases all follow. From Theorem 7.3 we have

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{(n-1)}(0),$$

whereas from Theorem 7.7 $\mathcal{L}\{t^m g(t)\} = (-1)^m \frac{d^m}{ds^m} G(s)$, where $\mathcal{L}\{g(t)\} = G(s)$. The main result of the theorem now follows by setting $g(t) = f^{(n)}(t)$ in this last result. ■

(i) $\mathcal{L}\{\exp(-t^2)\}$ and its connection with the error function

Laplace transform of the error function

We will use the differential equation satisfied by $y(t) = \exp(-t^2)$ to show that

$$\mathcal{L}\{\exp(-t^2)\} = \frac{1}{2}\sqrt{\pi}\exp(s^2/4)[1 - \operatorname{erf}(s/2)],$$

where

$$\operatorname{erf} s = \frac{2}{\sqrt{\pi}} \int_0^s \exp(-u^2)\,du$$

is a special function called the **error function**. The error function arises in the theory of heat conduction (see Section 7.3(f) and Chapter 18), in chemical diffusion processes, statistics, and elsewhere.

An attempt to find $\mathcal{L}\{\exp(-t^2)\}$ directly from the definition fails because the integral cannot be evaluated in terms of elementary functions, so some other method must be used. If we set $y(t) = \exp(-t^2)$, it is easily shown that $y(t)$ satisfies the first order variable coefficient equation

$$\frac{dy}{dt} + 2ty = 0,$$

subject to the initial condition $y(0) = \exp(0) = 1$.

Setting $\mathcal{L}\{y(t)\} = Y(s)$ and taking the Laplace transform of the differential equation gives

$$sY(s) - y(0) + 2\mathcal{L}\{ty(t)\} = 0.$$

However, $y(0) = 1$, and from result (i) of Theorem 7.15 (or directly from Theorem 7.7) $\mathcal{L}\{ty(t)\} = -Y'(s)$, so using these results in the preceding equation shows that the Laplace transform satisfies the differential equation

$$\frac{dY}{ds} - \frac{1}{2}sY = -\frac{1}{2}.$$

The integrating factor for this linear first order equation is $\mu(s) = \exp(-s^2/4)$, so after multiplication of the equation by $\mu(s)$ the result becomes

$$\frac{d}{ds}[\exp(-s^2/4)Y(s)] = -\frac{1}{2}\exp(-s^2/4).$$

Integrating over the interval $0 \le u \le s$ gives (after the introduction of the dummy variable u)

$$\int_0^s \frac{d}{du}[\exp(-u^2/4)Y(u)]du = -\frac{1}{2}\int_0^s \exp(-u^2/4)du,$$

or

$$\exp(-s^2/4)Y(s) - Y(0) = -\frac{1}{2}\int_0^s \exp(-u^2/4)du.$$

From the definition $Y(s) = \int_0^\infty e^{-st}\exp(-t^2)dt$, we find that $Y(0) = \int_0^\infty \exp(-t^2)dt$. The integral determining $Y(0)$ is a standard result, $\int_0^\infty \exp(-t^2)dt = \sqrt{\pi}/2$, so making use of this we find that

$$Y(s) = \frac{\sqrt{\pi}}{2}\exp(s^2/4)\left[1 - \frac{1}{\sqrt{\pi}}\int_0^s \exp(-u^2/4)du\right].$$

The change of variable $u = 2v$ brings this last result into the form

$$Y(s) = \frac{\sqrt{\pi}}{2}\exp(s^2/4)\left[1 - \frac{2}{\sqrt{\pi}}\int_0^{s/2} \exp(-v^2)dv\right].$$

If we now define the **error function as**

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-v^2)dv,$$

the Laplace transform $Y(s)$ becomes

$$Y(s) = \mathcal{L}\{\exp(-t^2)\} = \frac{\sqrt{\pi}}{2}\exp(s^2/4)[1 - \mathrm{erf}(s/2)].$$

The function erfc (x), defined as

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x),$$

is called the **complementary error function**, so in terms of this function the transform $Y(s)$ becomes

$$Y(s) = \frac{\sqrt{\pi}}{2}\exp(s^2/4)\mathrm{erfc}(s/2).$$

This method of determining the Laplace transform was successful because the differential equation satisfied by $Y(s)$ happened to be simpler than the differential equation satisfied by $y(t)$.

(ii) Laplace transform of the Bessel function $J_0(t)$ and the series expansion of $J_0(t)$

Laplace transform of a Bessel function

The following linear second order differential equation, called **Bessel's equation**,

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2)y = 0,$$

contains a parameter ν that is a constant. It has many applications, one of which is to be found in Chapter 18, where it enters into the solution of a vibrating circular membrane. The properties of its solutions are developed in some detail in Sections 8.6 and 8.7 of Chapter 8.

For each constant value ν, Bessel's equation has two linearly independent solutions denoted by $J_\nu(t)$ and $Y_\nu(t)$, called, respectively, Bessel functions of order ν of the first and second kind. We now use the Laplace transform to find $\mathcal{L}\{J_0(t)\}$, and then to find a power series expansion for $J_0(t)$ that will be obtained in a completely different way in Section 8.6. When $\nu = 0$, Bessel's equation reduces to

$$t\frac{d^2J_0}{dt^2} + \frac{dJ_0}{dt} + tJ_0 = 0,$$

and we will now find $\mathcal{L}\{J_0(t)\}$ subject to the initial condition $J_0(0) = 1$.

A second initial condition follows by setting $t = 0$ in the differential equation that gives $J_0'(0) = 0$, though this result will not be needed in what is to follow as the condition is implied later when the initial value Theorem 7.13 is used.

Taking the Laplace transform of Bessel's equation of order zero, setting $\mathcal{L}\{J_0(t)\} = Y(s)$, and using the results of Theorem 7.16, we obtain

$$-s^2Y'(s) - 2sY(s) + 1 + sY(s) - 1 - Y'(s) = 0,$$

and after simplification this shows that $Y(s)$ satisfies the first order differential equation

$$\frac{dY}{ds} + \frac{s}{s^2+1}Y(s) = 0.$$

Separating the variables and integrating gives

$$\int \frac{dY}{Y} = -\int \frac{s}{s^2+1} ds,$$

and so

$$Y(s) = \frac{C}{(s^2+1)^{1/2}}.$$

We now know the form of $Y(s)$, apart from the magnitude of the constant C. To find the constant we use the initial value theorem (Theorem 7.13), which shows that we must have

$$J_0(0) = \lim_{s\to\infty} [sY(s)],$$

but from the initial condition $J_0(0) = 1$, so

$$1 = \lim_{s\to\infty} \frac{sC}{(s^2+1)^{1/2}} = C,$$

and thus

$$\mathcal{L}\{J_0(t)\} = \frac{1}{(s^2+1)^{1/2}} \quad \text{for } s > 0.$$

This result can be used to obtain a series expansion for $J_0(t)$ by first writing it as

$$\mathcal{L}\{J_0(t)\} = \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-1/2},$$

and then expanding the result by the binomial theorem to obtain

$$\mathcal{L}\{J_0(t)\} = \frac{1}{s} - \frac{1}{2}\frac{1}{s^3} + \frac{3}{8}\frac{1}{s^5} - \frac{5}{16}\frac{1}{s^7} + \cdots.$$

Finally, taking the inverse Laplace transform of each term and adding the results, we arrive at the series expansion of $J_0(t)$:

$$J_0(t) = 1 - \frac{t^2}{4} + \frac{t^4}{64} - \frac{t^6}{2304} + \cdots.$$

If the general term in the expansion of $\frac{1}{s}(1 + \frac{1}{s^2})^{-1/2}$ is found, and the result is combined with entry 3 of Table 7.1, it is not difficult to show that $J_0(t)$ can be written as

$$J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}.$$

(iii) $\mathcal{L}\{\sin\sqrt{t}\}$

We now show how $\mathcal{L}\{\sin\sqrt{t}\} = Y(s)$ can be found from the differential equation satisfied by the function $\sin\sqrt{t}$, and how in this case a different form of argument from the one used in (ii) must be employed to determine the constant of integration

in the expression for $Y(s)$. It is easily seen that $y(t) = \sin \sqrt{t}$ is a solution of

$$4t \frac{d^2 y}{dt^2} + 2\frac{dy}{dt} + y = 0,$$

and clearly $y(0) = 0$. Writing $\mathcal{L}\{y(t)\} = Y(s)$, transforming the equation using result (iii) of Theorem 7.16, and incorporating the initial condition $y(0) = 0$ leads to the following first order differential equation for $Y(s)$:

$$\frac{dY}{ds} = \left(\frac{1 - 6s}{4s^2}\right) Y.$$

Integration of this variables separable equation gives

$$Y(s) = Cs^{-3/2} \exp[-1/(4s)],$$

so it only remains to determine the value of the constant C.

In this case the initial value theorem is of no help in determining C, so to accomplish this we return to the definition of the Laplace transform:

$$\mathcal{L}\{\sin \sqrt{t}\} = Y(s) = \int_0^\infty e^{-st} \sin \sqrt{t}\, dt.$$

The intuitive argument we now use can be made rigorous, but as the details of its justification are not appropriate here, they will be omitted. Inspection of the integrand shows that as $|\sin \sqrt{t}| \leq 1$ for all t, when s is large and positive the exponential function will only be significant close to the origin where the function $\sin \sqrt{t}$ can be approximated by $\sqrt{t}$. So for large s the integral can be approximated by

$$\begin{aligned} \mathcal{L}\{\sin \sqrt{t}\} &\approx \int_0^\infty e^{-st} t^{1/2} dt, \\ &= \frac{\Gamma(3/2)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}, \end{aligned}$$

where entry 4 of Table 7.1 has been used together with the result $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi}$ that will be proved later in Section 8.5 of Chapter 8.

Comparing the original expression for $Y(s)$ when s is large with this last result gives $C = \frac{1}{2}\sqrt{\pi}$, so

$$\mathcal{L}\{\sin \sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}} \exp[-1/(4s)], \quad \text{for } s > 0.$$

This form of argument used to determine the behavior of the integral as $s \to \infty$, where the approximation approaches arbitrarily close to the exact value as s increases, is called an *asymptotic* argument (see, for example, reference [3.3]).

(e) Two-Point Boundary Value Problems: Bending of Beams

boundary value problems and the bending of beams

The Laplace transform is ideally suited to the solution of initial value problems because of the way the initial values of a function enter into the Laplace transform of its derivatives. It can, however, also be used to solve certain types of two-point boundary value problems, as we now show. It will be helpful to use a simple physical example to illustrate the method of approach, so we will consider the case of a

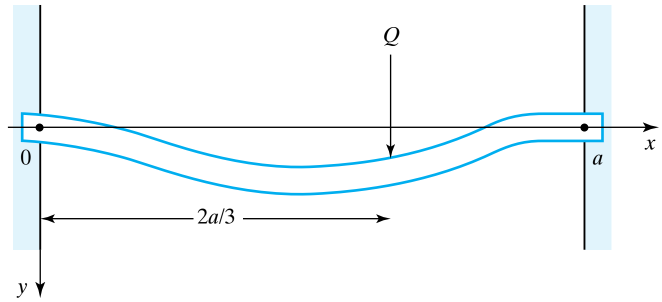


FIGURE 7.25 Clamped beam supporting a point load.

uniform horizontal beam of mass M and length a that is clamped at each end and supports a point load Q at a distance $2a/3$ from one end, as illustrated in Fig. 7.25.

The beam equation was introduced in Section 5.2(f) and is

$$EI\frac{d^4y}{dx^4} = w(x).$$

Here x is measured along the axis of the undeflected beam, $y(x)$ is the vertical deflection, E is the Young's modulus of the material of the beam, I is the second moment of the area of the beam about an axis normal to the x- and y-axes, and $w(x)$ is the transverse load per unit length of the beam, which in this case is an isolated point mass Q located at $x = 2a/3$. The boundary conditions for a clamped beam are

$$y(0) = y'(0) = 0 \quad \text{and} \quad y(a) = y'(a) = 0,$$

because neither deflection nor bending can occur at the ends, so both $y(x)$ and $y'(x)$ vanish at $x = 0$ and $x = a$.

The function $w(x)$ can be expressed as

$$w(x) = \frac{M}{a} + Q\delta(x - 2a/3), \quad \text{for } 0 \le x \le a,$$

where the point load Q is represented by the delta function that only makes a contribution at $x = 2a/3$.

Transforming the equation, setting $\mathcal{L}\{y(x)\} = Y(s)$, and this time writing x in place of t, because it is conventional to denote a length by x, we find

$$EI[s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)] = \mathcal{L}\{w(x)\}.$$

However,

$$\mathcal{L}\{w(x)\} = \frac{M}{as} + Qe^{-2as/3},$$

so using this in the preceding equation, incorporating the two known initial conditions $y(0) = y'(0) = 0$, and rearranging terms, we find that

$$Y(s) = \frac{M}{aEI}\frac{1}{s^5} + \frac{Q}{EI}\frac{e^{-2as/3}}{s^4} + \frac{1}{s^3}y''(0) + \frac{1}{s^4}y'''(0).$$

Taking the inverse Laplace transform of this expression gives

$$y(x) = \frac{M}{24aEI}x^4 + \frac{Q}{6EI}(x - 2a/3)^3H(x - 2a/3) + \frac{1}{2}x^2y''(0) + \frac{1}{6}x^3y'''(0).$$

We must now solve for the unknown initial conditions $y''(0)$ and $y'''(0)$ by requiring this expression to satisfy the two remaining boundary conditions at $x = a$, namely, $y(a) = y'(a) = 0$. The condition $y(a) = 0$ gives

$$0 = \frac{Ma}{4EI} + \frac{Qa}{27EI} + 3y''(0) + ay'''(0),$$

and the condition $y'(a) = 0$ gives

$$0 = \frac{Ma^2}{6EI} + \frac{Q}{18EI} + y''(0) + \frac{1}{2}ay'''(0),$$

so solving for $y''(0)$ and $y'''(0)$, we obtain

$$y''(0) = \frac{a}{108EI}(9M + 8Q) \quad \text{and} \quad y'''(0) = -\frac{1}{54EI}(27M + 14Q).$$

The required solution is then given by

$$y(x) = \frac{M}{24aEI}x^4 + \frac{Q}{6EI}(x - 2a/3)^3 H(x - 2a/3) + \frac{a}{216EI}(9M + 8Q)x^2 - \frac{1}{324EI}(27M + 14Q),$$

for $0 \leq x \leq a$.

This same form of approach can be used for other two-point boundary value problems, but its success depends on the ability to solve for the unknown initial values in terms of the given boundary conditions.

(f) An Application of the Laplace Transform to the Heat Equation

The Laplace transform can also be used to solve certain types of partial differential equation, involving two or more independent variables. Although the solution of partial differential equations (PDEs) forms the topic of Chapter 18, it will be instructive at this early stage to introduce a simple example that illustrates how the transform can be used for this purpose, and the way the result of Section 7.3d(i) enters into the solution.

a first encounter with a partial differential equation: the heat equation

The **one-dimensional heat equation** is the partial differential equation

$$\frac{1}{\kappa}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2},$$

where $T(x, t)$ is the temperature in a one-dimensional heat-conducting solid at position x at time t, and κ is a constant that describes the thermal conductivity property of the solid. This is a *partial* differential equation because it is a differential equation that involves the partial derivatives of the dependent variable $T(x, t)$. The physical situation modeled by this equation can be considered to be a semi-infinite slab of metal with a plane face on which the origin of the x-axis is located, with the positive half of the axis directed into the slab. This situation is illustrated in Fig. 7.26.

We will consider the situation where for $t < 0$ all of the metal in the slab is at the temperature $T = 0$ and then, at time $t = 0$, the plane face of the slab is suddenly brought up to and maintained at the constant temperature $T = T_0$. The problem is to find the temperature inside the slab on any plane $x =$ constant at any time $t > 0$, knowing that physically the temperature must remain finite for all $x > 0$ and $t > 0$.

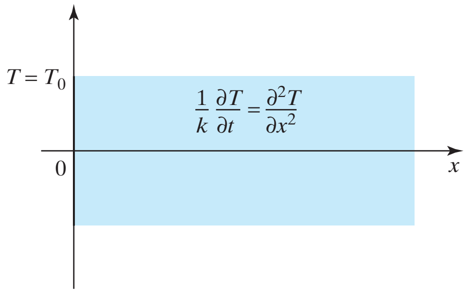


FIGURE 7.26 A semi-infinite metal slab.

The approach will be to take the Laplace transform of the dependent variable $T(x, t)$ in the heat equation with respect to the time t, as a result of which an ordinary differential equation with x as its independent variable will be obtained for the transformed variable that will then depend on both the Laplace transform variable s and x. After this ordinary differential equation has been solved for the transformed variable, the inverse Laplace transform will be used to recover the time variation, and so to arrive at the required solution as a function of x and t.

Before proceeding with this approach we notice first that if the Laplace transform is applied to the independent variable t in the function of two variables $T(x, t)$, the variable x will behave like a constant. Consequently, the rules for transforming derivatives of functions of a single independent variable also apply to a function of two independent variables. So, using the notation $\overline{T}(x, s) = {}_t\mathcal{L}\{T(x, t)\}$ to denote the Laplace transform of $T(x, t)$ with respect to the time t, it follows directly from the formula for the transform of a derivative in (9a) that

$$_t\mathcal{L}\{\partial T(x, t)\} = s\overline{T}(x, s) - T(x, 0).$$

To proceed further we must now use the condition that at time $t = 0$ the material of the slab is at zero temperature, so $T(x, 0) = 0$, as a result of which

$$_t\mathcal{L}\{\partial T(x, t)/\partial t\} = s\overline{T}(x, s).$$

Next, as x is regarded as a constant, we have

$$_t\mathcal{L}\{\partial^2 T(x, t)/\partial x^2\} = \frac{\partial^2 \overline{T}(x, s)}{\partial x^2}.$$

Using these results when taking the Laplace transform of the heat equation with respect to t, and making use of the linearity property of the transform, gives

$$s\overline{T}(x, s) = \kappa \frac{d^2 \overline{T}(x, s)}{dx^2},$$

where we now use an ordinary derivative with respect to x because in this differential equation s appears as a parameter so x can be considered to be the only independent variable. When the differential equation is written

$$\overline{T}'' - \frac{s}{\kappa}\overline{T} = 0,$$

using a prime to denote a derivative with respect to x, it is seen to have the general solution

$$\overline{T}(x, s) = A\exp\left[\sqrt{\frac{s}{\kappa}}x\right] + B\exp\left[-\sqrt{\frac{s}{\kappa}}x\right].$$

As a Laplace transform must vanish in the limit $s \to +\infty$, we must set $A = 0$, so the Laplace transform of the temperature is seen to be given by

$$\overline{T}(x, s) = B\exp\left[-\sqrt{\frac{s}{\kappa}}x\right].$$

In this case, the rejection of the term with the positive exponent in the general solution for $\overline{T}(x, s)$ corresponds to the physical requirement that the temperature remain finite for $x > 0$ and $t > 0$.

To determine B we now make use of the boundary condition on the plane face of the slab that requires $T(0, t) = T_0$, from which it follows that ${}_t\mathcal{L}\{T(0, t)\} = T_0/s$. Thus, the Laplace transform of the solution with respect to the time t is seen to be

$$\overline{T}(x, s) = \frac{T_0}{s}\exp\left[-\sqrt{\frac{s}{\kappa}}x\right].$$

To recover the time variation from this Laplace transform it is necessary to find ${}_t\mathcal{L}^{-1}\{\overline{T}(x, s)\}$. As $\overline{T}(x, s)$ is not the Laplace transform of an elementary function listed in our table of transform pairs, the solution $T(x, t)$ must be found by means of the Laplace inversion integral. In Chapter 16 on the Laplace inversion integral, it is shown in Example 16.6 that

$${}_t\mathcal{L}^{-1}\{e^{-k\sqrt{s}}\} = \frac{k}{2\sqrt{\pi t^3}}\exp\left\{-\frac{k^2}{4t}\right\}.$$

So, setting $k = x/\kappa^2$ in this result and using it with Theorem 7.11 to invert the Laplace transform $\overline{T}(x, s)$ shows that the solution is

$$\boxed{T(x, t) = T_0\,\mathrm{erfc}\left\{\frac{x}{2\sqrt{\kappa t}}\right\},} \qquad \text{for } x > 0,\ t > 0.$$

The use of integral transforms is discussed in reference [4.4].

Summary

The Laplace transform has been applied to systems of differential equations, and the results extended to systems in matrix form. Various applications have been made to some useful variable coefficient ordinary differential equations, and to the important partial differential equation that describes one-dimensional unsteady heat flow.

EXERCISES 7.3

(a) Exercises involving systems of equations

1. Solve

$$x' + 5x - 2y = 1 \quad \text{and} \quad y' - 5x + 2y = 3$$
$$\text{given } x(0) = 0,\ y(0) = 2.$$

2. Solve

$$x' - x - y = \cos t \quad \text{and} \quad y' + x + y = \cos t$$
$$\text{given } x(0) = 1,\ y(0) = 1.$$

3. Solve

$$x' + x + y = 2 \quad \text{and} \quad y' + x - y = 1$$
$$\text{given } x(0) = -1,\ y(0) = 1.$$

4. Solve

$$x' + x + 2y = e^{-t} \quad \text{and} \quad y' + 2x + y = 1 \quad \text{given}$$
$$x(0) = 0,\ y(0) = 0.$$

5. Solve

$$x' - x + 3y = 1 + t \quad \text{and} \quad y' + x - y = 2 \quad \text{given}$$
$$x(0) = 2,\ y(0) = -2.$$

6. Solve

$$x' + x + y = \sin 2t \quad \text{and} \quad y' + x - y = 1 \quad \text{given} \quad x(0) = 0,\ y(0) = 0.$$

7. Solve

$$x' + x - z = 1, \quad y' - x + y = 1, \quad z' + y - x = 0, \quad \text{given that } x(0) = 1,\ y(0) = 0,\ z(0) = 1.$$

8. Solve

$$x' + x - y = 1, \quad y' - y + 2z = 0, \quad z' + x - y = \sin t, \quad \text{given } x(0) = 1,\ y(0) = 0,\ z(0) = 2.$$

9. Solve

$$x' - z = e^t, \quad y' - z = 2, \quad z' - x = 1, \quad \text{given } x(0) = 0,\ y(0) = 1,\ z(0) = 0.$$

10. Solve

$$x' + z = 3, \quad y' + x = 1, \quad z' - x = \sin t, \quad \text{given } x(0) = 1,\ y(0) = 0,\ z(0) = 1.$$

(b) Exercises involving e^{tA}

In Exercises 11 through 24 find e^{tA} for the given matrix **A**.

11. $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$.

12. $\mathbf{A} = \begin{bmatrix} -2 & 4 \\ 3 & 2 \end{bmatrix}$.

13. $\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 2 & -1 \end{bmatrix}$.

14. $\mathbf{A} = \begin{bmatrix} 3 & 7 \\ 3 & -1 \end{bmatrix}$.

15. $\mathbf{A} = \begin{bmatrix} 4 & -5 \\ 4 & 0 \end{bmatrix}$.

16. $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 3 & -1 \end{bmatrix}$.

17. $\mathbf{A} = \begin{bmatrix} 2 & -4 \\ 1 & 2 \end{bmatrix}$.

18. $\mathbf{A} = \begin{bmatrix} -2 & 3 \\ 5 & 0 \end{bmatrix}$.

19. $\mathbf{A} = \begin{bmatrix} 6 & -1 \\ 0 & 6 \end{bmatrix}$.

20. $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$.

21. $\mathbf{A} = \begin{bmatrix} -2 & 4 \\ 0 & -2 \end{bmatrix}$.

22. $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 3 & 0 \end{bmatrix}$.

23. $\mathbf{A} = \begin{bmatrix} 5 & 10 & 7 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix}$.

24. $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

(c) Exercises involving the weighting function

In Exercises 26 through 32 find the weighting function when a single equation is involved, and the four weighting functions when a pair of equations is involved. Use the weighting function(s) to solve the given differential equation(s).

25. Show that if the initial conditions for equation (24) are $y(0) = y_0,\ y'(0) = y_1, \ldots, y^{(n-1)} = y_{n-1}$, the solution can be written in the form

$$y(t) = \int_0^t w(\tau)[y_0(t-\tau) - h(t-\tau)]d\tau.$$

Here $y_0(t)$ is the solution of the equation with the initial conditions $y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0$, and $h(t) = \{H(s)/G(s)\}$, with $H(s)$ the polynomial produced by the nonvanishing initial values of the derivatives, so that the transformed equation corresponding to (26) becomes

$$G(s)Y(s) + H(s) = F(s).$$

26. $y'' - 4y' + 3y = \cos t$, given $y(0) = 0$ and $y'(0) = 0$.

27. $y'' + 2y' + 2y = e^{2t}$, given $y(0) = 0$ and $y'(0) = 0$.

28. $y'' + 4y' + 13y = \cos 2t$, given $y(0) = 0$ and $y'(0) = 0$.

29. $y'' + 6y' + 5y = e^{-t}$, given $y(0) = 0$ and $y'(0) = 0$.

30. Use the result of Exercise 25 to solve

$$y'' - 2y' - 3y = 1 + \sin t, \quad \text{given } y(0) = 1 \quad \text{and} \quad y'(0) = -1.$$

31. $x' - 3x + 2y = e^{-t}, \quad y' + 3x - 4y = 3, \quad$ with $x(0) = y(0) = 0$.

32. $x' + 2x - y = \sin t, \quad y' - 2x + y = 2, \quad$ with $x(0) = y(0) = 0$.

(d) Differential equations with polynomial coefficients

33. Use the fact that $y(x) = \sin ax$ satisfies the differential equation

$$y'' + a^2 y = 0 \quad \text{with } y(0) = 0,\ y'(0) = a$$

to derive $\mathcal{L}\{\sin ax\}$ from the differential equation.

34. Use the fact that $y(x) = 1 - \cos ax$ satisfies the differential equation

$$y'' + a^2 y = a^2 \quad \text{with } y(0) = 0,\ y'(0) = 0$$

to derive $\mathcal{L}\{1 - \cos ax\}$ from the differential equation.

35.* The **Laguerre equation**

$$xy'' + (1-x)y' + ny = 0,$$

with $n = 0, 1, 2, \ldots$ a parameter, has polynomial solutions $y(x) = L_n(x)$ called **Laguerre polynomials**. These polynomials are used in many branches of mathematics and physics, and also in connection with numerical integration. By taking the Laplace transform of the differential equation find $\mathcal{L}\{L_n(x)\}$ and hence show that

$$L_4(x) = 24 - 96x + 72x^2 - 16x^3 + x^4.$$

36.* The **Hermite equation**

$$y'' - 2xy' + 2ny = 0,$$

with $n = 0, 1, 2, \ldots$ a parameter, has polynomial solutions $y(x) = H_n(x)$ called **Hermite polynomials**. Like the Laguerre polynomials, these polynomials are also used in mathematics and physics, and in connection with numerical integration. By transforming the equation and using the initial conditions $y(0) = H_4(0) = 12$ and $y'(0) = 0$, find $\mathcal{L}\{H_4(x)\}$, and hence show that

$$H_4(x) = 16x^4 - 48x^2 + 12.$$

37.* The Bessel function $y(x) = J_0(ax)$ satisfies the differential equation

$$xy'' + y' + axy = 0$$

subject to the initial conditions $y(0) = J_0(0) = 0$. Derive $\mathcal{L}\{J_0(ax)\}$ from the differential equation and confirm the result by using $\mathcal{L}\{J_0(x)\} = 1/(s^2+1)^{1/2}$ in conjunction with the scaling theorem.

38.* The Bessel function $y(x) = J_1(x)$ satisfies the differential equation

$$x^2y'' + xy' + (x^2 - 1)y = 0 \quad \text{with } J_1(0) = 0 \text{ and } J_1{}'(0) = 1/2.$$

By taking the Laplace transform of the differential equation show that $\mathcal{L}\{J_1(x)\} = C\{1 - s/(s^2+1)^{1/2}\}$, and deduce that $C = 1$.

(e) Exercises involving two-point boundary value problems

39. Solve $x'' + x = \sin 2t$ with $x(0) = 0$ and $x(\pi/2) = 1$.

40. Using the notation of Section 7.3(e), solve the beam equation

$$EI\frac{d^4y}{dx^4} = w(x)$$

for the uniform cantilevered beam of mass M and length a shown in Fig. 7.27, where a point mass Q is located at a distance $a/3$ from the clamped end. The boundary conditions to be used are

$$y(0) = y'(0) = 0 \quad \text{and} \quad y''(a) = y'''(a) = 0.$$

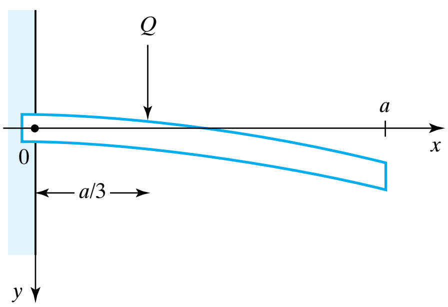


FIGURE 7.27 Cantilevered beam with a point load.

41. Using the notation of Section 7.3(e), solve the beam equation

$$EI\frac{d^4y}{dx^4} = w(x)$$

for the uniform beam of mass M and length a with clamped ends shown in Fig. 7.28, where a point mass Q is located at a distance $3a/4$ from the left-hand end. The boundary conditions to be used are

$$y(0) = y'(0) = 0 \quad \text{and} \quad y(a) = y'(a) = 0.$$

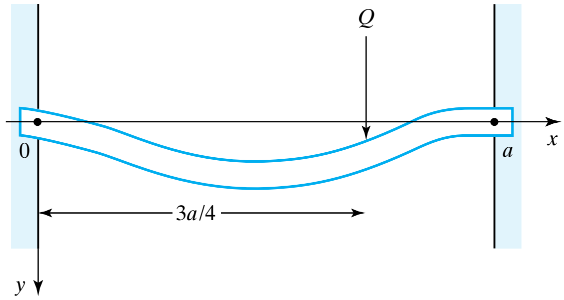


FIGURE 7.28 Supported beam with clamped ends and a point load.

42. Using the notation of Section 7.3(e), solve the beam equation

$$EI\frac{d^4y}{dx^4} = w(x)$$

for the uniform beam of mass M and length a shown in Fig. 7.29 that is clamped at the end $x = 0$ and supported at the end $x = a$, where a point mass Q is located at a distance $a/4$ from the right-hand end. The boundary conditions to be used are

$$y(0) = y'(0) = 0 \quad \text{and} \quad y(a) = y''(a) = 0.$$

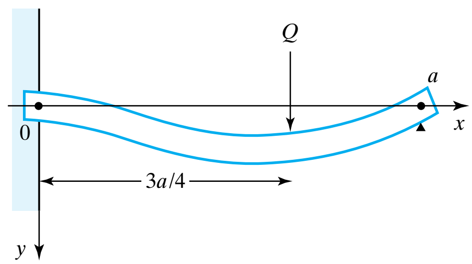


FIGURE 7.29 Beam clamped at one end and supported at the other with a point load.

(f) Physical problems to be solved by computer algebra

43. In an R–L–C circuit the current $i(t)$ and charge $q(t)$ resulting from a constant voltage E_0 applied at time

$t = 0$, when $i(0) = 0$ and $q(0) = 0$, are determined by the equations

$$L\frac{di}{dt} + Ri + \frac{q}{C} = E_0 \quad \text{and} \quad i = \frac{dq}{dt}.$$

Find $i(t)$, and comment on its form depending on the sign of $R^2C - 4L$. Choose representative values of R, L, C corresponding to each of the foregoing cases and plot $i(t)$ in a suitable interval $0 \le t \le T$.

44. Figure 6.10 in Section 6.3 illustrates three particles of equal mass joined by identical springs that oscillate in a straight line, with each end of the system clamped. In a representative case, the nondimensional equations determining the magnitudes of the displacements $y_1(t)$, $y_2(t)$, and $y_3(t)$ are

$$3\frac{d^2y_1}{dt^2} = y_2 - 2y_1 + y_3, \quad 3\frac{d^2y_2}{dt^2} = y_3 - 2y_2 + y_1,$$
$$3\frac{d^2y_3}{dt^2} = y_1 - 2y_3 + y_2.$$

Find $y_1(t)$, $y_2(t)$, and $y_3(t)$ given that $y_1(0) = 1$, $y_1'(0) = 0$, $y_2(0) = 2$, $y_2'(0) = 1$, $y_3(0) = 3$, $y_3'(0) = 0$.

45. If, similar to the example in Section 7.3(a), an irreversible reaction converts a molecule of chemical A into a molecule of chemical D, via molecules of chemicals B and C, the governing equations in terms of the respective reaction rates k_1, k_2, and k_3 are

$$\frac{dx}{dt} = -k_1x, \quad \frac{dy}{dt} = k_1x - k_2y, \quad \text{and} \quad \frac{dz}{dt} = k_2y - k_3z,$$

where x, y, and z are the number of molecules of A, B, and C present at time t. If Q molecules of A are present at time $t = 0$, the number of molecules of D present at time t is $w(t) = Q - x(t) - y(t) - z(t)$. Find $w(t)/Q$ as a function of t given that $k_1 = 2$, $k_2 = 3$, and $k_3 = 3$, and plot the result for $0 \le t \le 5$. Find the percentage of chemical A that has been transformed into chemical D at the instants of time $t = 1, 2$, and 3.

46. In the following nondimensional equations, $x(t)$ and $y(t)$ represent the magnitudes of the currents flowing in the primary and secondary windings of a transformer, when initially $x(0) = 0$, $y(0) = 0$ and at time $t = 0$ the primary winding is subjected to an exponentially decaying voltage of magnitude e^{-t}:

$$\frac{dx}{dt} + \frac{1}{3}\frac{dy}{dt} + 3x = e^{-t}, \quad \frac{dx}{dt} + 3\frac{dy}{dt} + 9y = 0.$$

Find $x(t)$ and $y(t)$, and by plotting the magnitudes of the currents show that $x(t)$ is always positive and after peaking decays to zero, while $y(t)$ is initially negative, but after becoming positive it decays to zero faster than $x(t)$.

7.4 The Transfer Function, Control Systems, and Time Lags

The study of engineering systems of all types whose behavior is determined by *linear* ordinary differential equations is often carried out by examining what is called the system **transfer function**. Typically, a system is governed by a linear nth order constant coefficient ordinary differential equation whose solution or **output**, also called the **response** of the system, we will denote by $u_0(t)$ and whose forcing function, or **input**, is a known function we will denote by $u_i(t)$, where t is the time.

A typical example of a simple system has already been encountered in Fig. 6.2, where the spring-mounted and damped vibrating machine has an input $F(t)$ and an output $y(t)$ that are related by

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = F(t).$$

An nth order system may be governed by the equation

$$a_n\frac{d^nu_0}{dt^n} + a_{n-1}\frac{d^{n-1}u_0}{dt^{n-1}} + \cdots + a_0u_0 = u_i,$$

which can be represented graphically as in Fig. 7.30, where F[.] is the differential operator

$$F[.] \equiv a_n\frac{d^n}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_0. \tag{34}$$

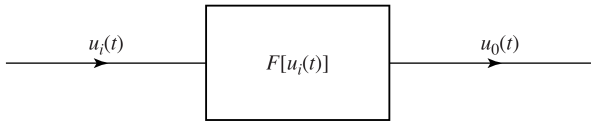


FIGURE 7.30 Block-diagram representation of equation (34).

More generally, in linear systems the input itself may be the solution of another linear differential equation, in which case the system relating the response $u_0(t)$ to the input $u_i(t)$ becomes

$$a_n \frac{d^n u_0}{dt^n} + a_{n-1}\frac{d^{n-1}u_0}{dt^{n-1}} + \cdots + a_0 u_0 = b_m \frac{d^m u_i}{dt^m} + b_{m-1}\frac{d^{m-1}u_i}{dt^{m-1}} + \cdots b_0 u_i, \tag{35}$$

where $n \geq m$ and the coefficients a_r and b_s are constants.

The **transfer function** of a system is defined as the quotient of the Laplace transforms of the system output and the system input, when all of the initial conditions are taken to be *zero*. This last condition means that when the Laplace transform is used to transform a differential equation we may set $\mathcal{L}\{d^r u/dt^r\} = s^r U(s)$. So, after transforming (35), we obtain

$$(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0)U_0(s) = (b_m s^m + b_{m-1}s^{m-1} + \cdots b_0)U_i(s), \tag{36}$$

where $U_0(s) = \mathcal{L}\{u_0(t)\}$ and $U_i(s) = \mathcal{L}\{u_i(t)\}$. The transfer function $G(s) = U_0(s)/U_i(s)$ becomes the rational function of the transform variable s

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots a_0}. \tag{37}$$

Let us now set $G(s) = N(s)/D(s)$, where $N(s)$ is the polynomial in s of degree m in the numerator of $G(s)$, and $D(s)$ is the polynomial in s of degree n in the denominator. The polynomial $D(s)$ is called the **characteristic polynomial** of the system, and $D(s) = 0$ is called the **characteristic equation** of the system. The **order** of the system in (37) is the degree n of the polynomial $D(s)$.

As the coefficients of $D(s)$ are real, it follows that the roots of the characteristic equation, called the **poles** of the transfer function $G(s)$, either are all real or, if complex, they must occur in complex conjugate pairs. When $G(s)$ is expressed in partial fraction form, this last observation implies that the system will be **stable** provided all the roots of the characteristic equation have negative real parts. Here, by **stability**, we mean that any bounded input to a system that is stable will result in an output that is also bounded for all time, and this will be the case when every root of $D(s) = 0$ has a negative real part. The requirement that $n \geq m$ imposed on (35) is necessary in order to prevent unbounded behavior of the output caused by the occurrence of delta functions.

It is important to recognize that systems describing quite different physical phenomena can have the *same* transfer function, so transfer functions provide a means of examining a class of similar systems independently of their physical origin. It follows that for any given input with Laplace transform $U_i(s)$, the Laplace transform of the output $U_0(s)$ is given by

$$U_0(s) = G(s)U_i(s). \tag{38}$$

The time variation of the output of the system then follows by taking the inverse Laplace transform of (38).

EXAMPLE 7.36

Find the transfer function of the system with input $u_i(t)$ and output $u_0(t)$ described by

$$4\frac{d^2u_0(t)}{dt^2} + 16\frac{du_0(t)}{dt} + 25u_0(t) = 3\frac{du_i(t)}{dt} + 2u_i(t),$$

and show it is stable.

Solution Taking the Laplace transform of the governing equation and assuming all initial conditions to be zero gives

$$(4s^2 + 16s + 25)U_0(s) = (3s + 2)U_i(s),$$

so the system transfer function is

$$G(s) = \frac{U_0(s)}{U_i(s)} = \frac{3s+2}{4s^2 + 16s + 25}.$$

The system is of order 2, and its characteristic equation is

$$4s^2 + 16s + 25 = 0.$$

The characteristic equation has the roots $s_1 = -2 - \frac{3}{2}i$ and $s_2 = -2 + \frac{3}{2}i$, so as their real parts are negative, the system is stable. ■

Systems that compare the difference between an input and an output, and attempt to reduce the difference to zero to make the output *follow* the input, are called **control systems**. A typical example is a temperature control system for a chemical reactor in which the temperature is required to remain constant, but where as the reaction progresses heat is released at variable rates, causing cooling to become necessary.

A simple control system is illustrated in Fig. 7.31, where F is the system differential equation. The idea here is that an input u_i is compared with the output u_0, called the **feedback**, and the difference $\varepsilon = u_i - u_0$, called the **error signal**, is then used as an input to system F. The result is that $u_0 = u_i$ when $\varepsilon = 0$. It is often necessary to modify the feedback by passing u_0 through another system G with output $v = G[u_0]$, and then to use the the difference $v - u_i$ to drive F. The reason for this is to improve the overall performance of a system, whose physical characteristics may be difficult to alter, by using an easily modified feedback to make the system more responsive and to reduce any tendency it may have for excessive oscillation.

EXAMPLE 7.37

A steering mechanism for a small boat comprises an input heading θ_i from the helm, an amplifier for the error signal, and a servomotor to drive the rudder with moment of inertia I that produces a resisiting torque proportional to the rate of change of the output angle θ_0. Derive the differential equation governing the system and find its transfer function given that the feedback is the unmodified output θ_0.

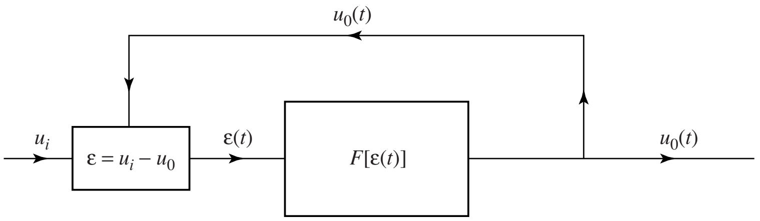


FIGURE 7.31 A typical feedback control system.

Solution If the resisting torque is $kd\theta_0/dt$ and the amplifier increases the magnitude of the error signal by a factor K, the system can be represented as in Fig. 7.31 with the governing differential equation

$$I\frac{d^2\theta_0}{dt^2} + k\frac{d\theta_0}{dt} = K(\theta_i - \theta_0).$$

Taking the Laplace transform of this equation gives

$$(Is^2 + ks + K)\mathcal{L}\{\theta_0\} = \mathcal{L}\{\theta_i\},$$

and so

$$\mathcal{L}\{\theta_0\} = \frac{1}{Is^2 + ks + K}\mathcal{L}\{\theta_i\}.$$

This result shows that the transfer function $G(s) = 1/(Is^2 + ks + K)$, so the system will be stable provided the roots of the characteristic equation $Is^2 + ks + K = 0$ have negative real parts. This will be the case since $I > 0$ and $K > 0$, but the steering will oscillate about the required heading if $4IK > k^2$.

As the design of the boat determines I and k, any improvement of the steering response can only be obtained by using a modified feedback signal instead of the direct feedback θ_0. ■

We close this section by mentioning an important consequence of the introduction of a delay into an equation governing the response of a system. Consider a vibrating system characterized by $y(t)$ in which instantaneous damping proportional to the velocity dy/dt occurs with coefficient of proportionality a_1, and where there is also present an additional time retarded damping of a similar type but with a time lag τ and a coefficient of proportionality a_2. Then, when a springlike restoring effect is present with constant of proportionality a_3, the governing equation takes the form

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_2\frac{dy(t-\tau)}{dt} + a_3y(t) = 0. \tag{39}$$

Because of the presence of the time-delayed derivative $dy(t-\tau)/dt$, an equation of this type is called a **differential-difference equation**.

If we now seek a solution of this equation by using the Laplace transform (or by seeking solutions of the form $y(t) = A\exp(\lambda t)$, where A and λ are constants) we arrive at a characteristic equation of the form

$$s^2 + a_1s + a_2s\exp(-\tau s) + a_3 = 0. \tag{40}$$

This is called an **exponential polynomial** in s, and its root will determine both the stability and response of the system.

Without going into detail, by using Rouche's theorem from complex analysis it is not difficult to prove that exponential polynomials have an infinite number of zeros. Consequently, the response of a system with a characteristic polynomial in the form of an exponential polynomial will only be stable if all of its zeros have negative real parts, and this can only be shown analytically. Methods exist that can be used to determine when all the zeros of such exponential polynomials have negative real parts. An interested reader will find a valuable discussion of this subject in Section 13 of *Differential-Difference Equations* by R. Bellman and K. Cooke, published by Academic Press in 1963.

It is necessary to ask in what way the infinite number of zeros of an exponential polynomial of degree n approximate the n zeros of the ordinary polynomial of

degree n when time lags are absent. This is a simpler question, and it can be answered by appeal to Hurwitz's theorem from complex analysis, though again the arguments used go beyond this first account of the subject.

A result on exponential polynomials

Let $P_\tau(s)$ be an exponential polynomial of degree n in s with a time lag τ, and let $P_0(s)$ be the corresponding constant coefficient polynomial when $\tau = 0$. Then, as $\tau \to 0$, so each of the n zeros s_i of $P_0(s)$ is approached arbitrarily closely by a number of zeros of $P_\tau(s)$ equal in number to its multiplicity, and the remaining infinite number of zeros of $P_\tau(s)$ can be made to lie outside a circle of arbitrarily large radius centered on the origin.

As this result says nothing about how the zeros move as $\tau \to 0$, it is possible for the system to be stable when τ lies in certain intervals and unstable otherwise.

EXERCISES 7.4

1. Find the transfer function for each of the following systems. Determine the order of each system and find which is stable.

(a) $$\frac{d^3u_0}{dt^3} + 3\frac{d^2u_0}{dt^2} + 16\frac{du_0}{dt} - 20u_0 = 2\frac{d^2u_i}{dt^2} + \frac{du_i}{dt} - 6u_i.$$

(b) $$\frac{d^3u_0}{dt^3} + 4\frac{d^2u_0}{dt^2} + 14\frac{du_0}{dt} + 20u_0 = 6\frac{d^2u_i}{dt^2} - 13\frac{du_i}{dt} + 6u_i.$$

(c) $$9\frac{d^2u_0}{dt^2} + 6\frac{du_0}{dt} + 10u_0 = 6\frac{d^2u_i}{dt^2} + 5\frac{du_i}{dt} - 6u_i.$$

2.* For safety reasons, a control system is often duplicated, with the sensors for each system located in different positions, and in such cases the possibility of interaction between the control systems must be considered. A typical case is illustrated in Fig. 7.32, where two identical control systems are shown between which there is assumed to be linear **cross-coupling** of the error signals. This means that the respective actuating error signals are $\varepsilon_1' = a_{11}\varepsilon_1 + a_{12}\varepsilon_2$ and $\varepsilon_2' = a_{21}\varepsilon_1 + a_{22}\varepsilon_2$, with the coefficients a_{ij} constants. Derive and discuss the equations governing the response of the system when

$$F(u_0) = \frac{d^2u_0}{dt^2} + 2\zeta\Omega\frac{du_0}{dt} + \Omega^2u_0,$$

with $\zeta > 0$ and $\Omega > 0$.

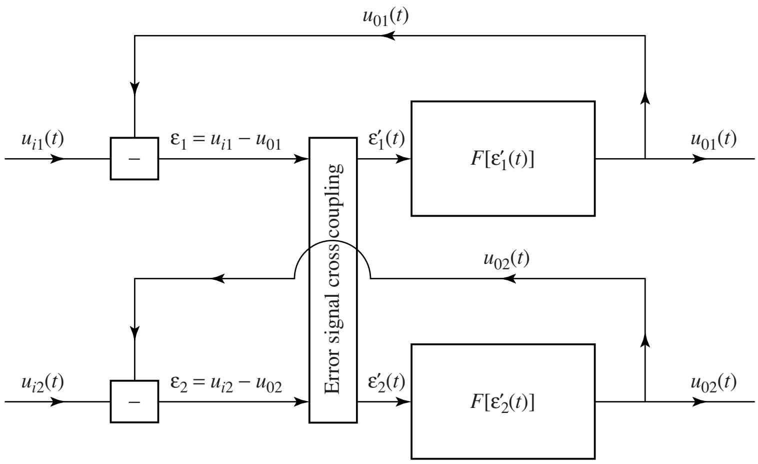
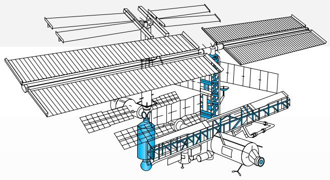


FIGURE 7.32 Two interacting control systems.



CHAPTER 7

TECHNOLOGY PROJECTS

The purpose of these projects is to use a computer algebra differential equation solver to find the analytical solutions of initial value problems involving linear constant coefficient differential equations, some of which contain either the Dirac delta function or the Heaviside step function. As all the initial conditions are given at $t = 0$, the Laplace transform can also be used to solve these problems.

Project 1

Solving a Third Order Initial Value Problem

Use a computer algebra Laplace solver to solve the initial value problem

$$x''' + 2x'' - x' - 2x = e^{-t}\sin t, \quad \text{with } x(0) = 1,\ x'(0) = -1, \text{ and } x''(0) = 0.$$

Verify the result by using computer algebra (a) to take the Laplace transform of the equation, (b) to find the Laplace transform $X(s)$ of the solution, and (c) to invert the transform to find $x(t)$.

Project 2

Solving an Equation with the Heaviside Step Function in the Nonhomogeneous Term

Use a computer algebra Laplace solver to solve the initial value problem

$$x'' + 3x' + 2x = \{H(t-1) - H(t-2)\}t, \quad \text{with } x(0) = 1,\ x'(0) = -1.$$

Verify the result by using computer algebra (a) to take the Laplace transform of the equation, (b) to find the Laplace transform $X(s)$ of the solution, and (c) to invert the transform to find $x(t)$. Plot the solution for $0 \le t \le 6$.

Project 3

Solving an Equation with the Dirac Delta Function in the Nonhomogeneous Term

Use a computer algebra Laplace solver to solve the initial value problem

$$x'' + 3x' + 2x = 3e^{-t} + \delta(t-2), \quad \text{with } x(0) = 1,\ x'(0) = 2.$$

Verify the result by using computer algebra (a) to take the Laplace transform of the equation, (b) to find the Laplace transform $X(s)$ of the solution, and (c) to invert the transform to find $x(t)$.

Project 4

Solving a System

Solve the initial value problem for the system

$$\frac{dx}{dt} = x(t) + 2y(t) + 3, \quad \frac{dy}{dt} = 1 - x(t) + y(t), \quad \text{with } x(0) = 1,\ y(0) = 0.$$

Verify the result by using computer algebra (a) to take the Laplace transform of the system, (b) to solve for the Laplace transforms $X(s)$ and $Y(s)$ of $x(t)$ and $y(t)$, and then (c) to invert the transforms to find $x(t)$ and $y(t)$.

Project 5

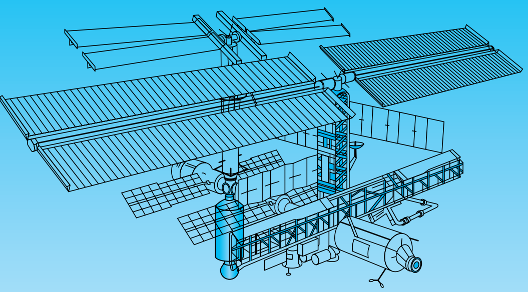
Examining the Properties of a Spring Damper System

In an experiment, a wheel of mass M is mounted vertically below a rigid plate to which it is attached by a spring with spring constant k and a damper whose resisting force is μ times the speed of its displacement. If at time t the vertical displacement of the wheel from its equilibrium position is $x(t)$, and a force $F(t)$ is applied to the wheel, its equation of motion is

$$M\frac{d^2x}{dt^2} + \mu\frac{dx}{dt} + kx = F(t).$$

Set $\Omega = (k/M)^{1/2}$, $\zeta = \frac{\mu}{2\sqrt{kM}}$, and assume the wheel is initially at rest, so that $x(0) = 0$ and $(dx/dt)_{t=0} = 0$. If a constant load $F(t) = F_0$ is suddenly applied to the wheel at the time $t = 0$, find an expression for $x(t)k/F_0$. Plot this expression for several values of ζ in the interval $0 < \zeta < 2$ and comment on the results.

CHAPTER 8



Series Solutions of Differential Equations, Special Functions, and Sturm–Liouville Equations

Linear second order variable coefficient equations arise in many applications, but only in a few special cases is it possible to express their general solution of a finite linear combination of elementary functions. As analytical, rather than purely numerical, information about solutions is often essential, some other way must be found to represent the solutions of such equations. The approach developed in this chapter involves seeking solutions of certain types of equation in the form of power series, and in other cases using an approach due to Frobenius that involves seeking solutions in the form of power series multiplied by a factor x^c, where c is not an integer. Applications are made to a number of typical linear variable coefficient equations, and then to the important Legendre, Chebyshev, and Bessel equations that lead in turn to Legendre and Chebyshev polynomials and to Bessel functions.

Two-point boundary value problems, called Sturm–Liouville systems, that are defined over an interval $a \le x \le b$ and contain a parameter λ are introduced. It is shown that their solutions only exist for an infinite number of special values of the parameter $\lambda_1, \lambda_2, \ldots,$ called the eigenvalues of the problem. Each solution $\varphi_n(x)$ corresponding to an eigenvalue λ_n is called an eigenfunction, and the eigenfunctions are shown to have the special property of orthogonality with respect to a function $w(x)$ called the weight function. This means that if the set of eigenfunctions is $\{\varphi_n(x)\}_{n=1}^{\infty}$, the integral $\int_a^b \varphi_m(x)\varphi_n(x)w(x)dx$ is positive when $n = m$ and zero when $n \ne m$. This property will be used extensively in Chapter 18 when solving partial differential equations.

Fundamental properties of eigenfunctions and eigenvalues are established for general Sturm–Liouville systems, after which a number of frequently occurring and important special cases are examined.

8.1 A First Approach to Power Series Solutions of Differential Equations

The solutions of many differential equations can be expressed in terms of elementary functions such as sine, cosine, exponential, and logarithm, all of whose mathematical properties are well known. When required, the analytical behavior of solutions that involve elementary functions can be explored by making use of

their familiar properties. Numerical solutions are obtained easily, either by using a pocket calculator to find the values of the elementary functions involved, or through the use of standard subroutines that form a part of all basic mathematical software packages. With either a pocket calculator or a software package, the method of calculating functional values is usually based on a series expansion of the function concerned.

Most differential equations cannot be solved in terms of elementary functions, yet some form of analytical solution is often needed rather than a purely numerical one, so the fundamental question that then arises is how to obtain a solution in the form of a series, when only the differential equation is known. It is the purpose of this chapter to answer this question, and in the process to show how the form of series solution obtained depends on what are called the *singular points* of the differential equation.

We begin our approach to this problem by showing how series solutions can be found for first and second order linear differential equations with initial conditions specified at $x = x_0$. The series we obtain will be in powers of $x - x_0$, and they will be said to be *expanded* about the point x_0. The first order linear differential equation will be assumed to be of the form

$$y' + p(x)y = r(x) \quad \text{with } y(x_0) = y_0, \tag{1}$$

and the second order linear differential equation will be assumed to be of the form

$$y'' + P(x)y' + Q(x)y = R(x) \quad \text{with } y(x_0) = y_0,\ y'(x_0) = y_1, \tag{2}$$

where the functions $p(x)$, $r(x)$, $P(x)$, $Q(x)$, and $R(x)$ can all be expanded as Taylor series about the point x_0.

analytic in a neighborhood

Functions with this property are said to be **analytic** in a **neighborhood** of the point x_0 or, more simply, to be **analytic** at x_0. The method to be developed will be seen to be capable of extension to a higher order linear differential equation in an obvious manner, provided only that the coefficients of y and its derivatives that are involved and the nonhomogeneous term are analytic at x_0.

how to find a power series solution

The approach is best illustrated by considering equation (1), and seeking a solution about x_0 of the form

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} y^{(n)}(x_0), \quad \text{with } y^{(n)}(x) = d^n y/dx^n. \end{aligned} \tag{3}$$

Setting $x = x_0$ in (1) gives

$$y^{(1)}(x_0) + p(x_0)y(x_0) = r(x_0),$$

but $y(x_0) = y_0$, so

$$\begin{aligned} y^{(1)}(x_0) &= r(x_0) - p(x_0)y(x_0) \\ &= r(x_0) - p(x_0)y_0. \end{aligned}$$

To determine $y^{(2)}(x)$ we differentiate equation (1) once with respect to x to obtain

$$y^{(2)}(x) + p^{(1)}(x)y(x) + p(x)y^{(1)}(x) = r^{(1)}(x),$$

where $p^{(1)}(x) = p'(x)$ and $r^{(1)}(x) = r'(x)$. Then, after setting $x = x_0$ and using the fact that $y^{(1)}(x_0) = r(x_0) - p(x_0)y_0$, we find that

$$y^{(2)}(x_0) = r^{(1)}(x_0) - p^{(1)}(x_0)y_0 - p(x_0)[r(x_0) - p(x_0)y_0].$$

Higher order derivatives $y^{(n)}(x_0)$ can be computed in similar fashion by repeated differentiation of the original differential equation coupled with the use of lower order derivatives that have already been determined. Once the values of $y^{(k)}(x_0)$ have been found for $k = 1, 2, \ldots, N$, for some given integer N, substitution into series (3) provides the required approximation to the power series solution of the initial value problem for the differential equation up to terms of order $(x - x_0)^N$. The existence and uniqueness of the solution are guaranteed by Theorem 5.2.

This method generates the Taylor series expansion of $y(x)$ about the point x_0 when $x_0 \neq 0$, and its Maclaurin series expansion when $x_0 = 0$, though these series are often simply called power series about $x_0 \neq 0$ and $x_0 = 0$, respectively.

EXAMPLE 8.1 Find the first five terms in the series solution of

$$y' + (1 + x^2)y = \sin x, \quad \text{with } y(0) = a.$$

Solution As the initial condition is specified at $x = 0$, the power series solution is an expansion about the origin and so is, in fact, a Maclaurin series. The functions $1 + x^2$ and $\cos x$ are analytic for all x, so the series expansion can certainly be found about the origin.

Setting $x = 0$ in the equation and substituting for the initial conditions shows that $y'(0) = y^{(1)}(0) = -a$. Differentiation of the differential equation gives

$$y^{(2)} + 2xy + (1 + x^2)y^{(1)} = \cos x,$$

where $y^{(2)} = y''$, so setting $x = 0$ this becomes

$$y^{(2)}(0) + y^{(1)}(0) = 1,$$

but $y^{(1)}(0) = -a$ and so $y^{(2)}(0) = 1 + a$. Repeating this process to find higher order derivatives leads to the results $y^{(3)}(0) = -(1 + 3a)$, $y^{(4)}(0) = 9a, \ldots$. Substituting these results into series (3) shows that, to terms of order x^4, the required solution takes the form

$$y(x) = a - ax + (1 + a)\frac{x^2}{2!} - (1 + 3a)\frac{x^3}{3!} + 9a\frac{x^4}{4!} + \cdots.$$

■

EXAMPLE 8.2 Find the first five terms in the series solution of

$$y' + 4xy = 3e^{x-1}, \quad \text{with } y(1) = 1.$$

Solution In this case the functions x and e^{x-1} are analytic for all x, but as the expansion is about $x = 1$, the power series solution that is obtained will be a Taylor series expansion about the point $x = 1$. Setting $x = 1$ in the differential equation and using the initial condition $y(1) = 1$ shows that $y^{(1)}(1) = -1$.

Differentiation of the differential equation gives

$$y^{(2)} + 4y + 4xy^{(1)} = 3e^{x-1},$$

so setting $x = 1$ and using the result $y^{(1)}(1) = -1$ shows that $y^{(2)}(1) = 3$.

Repeating this process leads to the results that $y^{(3)}(1) = -1$ and $y^{(4)}(1) = -29$, so substituting into (3) shows that the Taylor series expansion of the solution up to terms of order $(x-1)^4$ is

$$y(x) = 1 - (x-1) + \frac{3}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 - \frac{29}{24}(x-1)^4 + \cdots. \quad \blacksquare$$

This same method can be applied to a second order equation of the type shown in (2), though a more general approach will be developed later to deal with the case in which the first term is of the form $a(x)y''(x)$, and the expansion is about a point x_0 where $a(x_0) = 0$.

EXAMPLE 8.3 Find the terms up to x^5 in the series solution of

$$y'' + xy' + (1 - x^2)y = x \quad \text{with } y(0) = a,\ y'(0) = b.$$

Solution The coefficients x and $(1 - x^2)$ and the nonhomogeneous term x are analytic for all x, so as the initial data is given at $x = 0$, a Maclaurin series solution can be found.

Setting $x = 0$ in the equation and using the initial conditions $y(0) = a$ and $y'(0) = b$ gives $y^{(2)}(0) = -a$. Differentiating the differential equation we have

$$y^{(3)} + y^{(1)} + xy^{(2)} - 2xy + (1 - x^2)y^{(1)} = 1,$$

so setting $x = 0$ and using the results $y^{(2)}(0) = -a$ and $y^{(1)}(0) = b$ shows that $y^{(3)}(0) = 1 - 2b$. A repetition of this process leads to the results $y^{(4)}(0) = 5a$, $y^{(5)}(0) = 14b - 4, \ldots,$ so substituting into (3) shows that to terms of order x^5 the Maclaurin series expansion of the solution is

$$y(x) = a + bx - \frac{1}{2}ax^2 + \left(\frac{1-2b}{6}\right)x^3 + \frac{5a}{24}x^4 + \left(\frac{7b-2}{60}\right)x^5 + \cdots. \quad \blacksquare$$

Summary

Often a variable coefficient equation cannot be solved in terms of known functions, though some form of analytical solution is still required. This section has shown how to overcome this difficulty in some cases by finding a solution in terms of a power series expanded about a point of interest $x = a$. The method was seen to work provided the functions in the equation have Taylor series expansions about $x = a$. It will be shown later how to find series solutions in a systematic manner, and also how to generalize this approach to other types of equation.

EXERCISES 8.1

Find the first five terms in the power series solution of the following initial value problems.

1. $y' + (1 + x^2)y = x^2$, with $y(0) = 1$.
2. $2y' + xy = 1 - x$, with $y(0) = 2$.
3. $y' + (1 - 2x)y = x$, with $y(0) = -1$.
4. $4y' + (1 + x + x^2)y = x$, with $y(0) = 3$.
5. $y' + (x - 2x^2)y = 1$, with $y(0) = 1$.
6. $y' - 2xy = 1 - x$, with $y(0) = 2$.
7. $3y' + (1 - x^2)y = 1$, with $y(0) = 2$.
8. $y' + (1 + x)y = 1 + x^2$, with $y(0) = 1$.
9. $y'' - 2xy' + x^2y = 0$, with $y(0) = a,\ y'(0) = b$.
10. $2y'' + 2(1 + x)y' - y = 0$, with $y(0) = a,\ y'(0) = b$.

11. $(1+x^2)y'' + 3xy' + (1-x^2)y = 1 + x$, with $y(0) = a$, $y'(0) = b$.

12. $(1+3x^2)y'' + 2xy' + 2xy = 1$, with $y(0) = a$, $y'(0) = b$.

13. $y'' + 7y' + x^2 y = 0$, with $y(0) = a$, $y'(0) = b$.

14. $xy'' + (1+x)y' + xy = b$, with $y(0) = a$, $y'(0) = 0$.

15. $2y'' + 3x^2 y' + (1-x^2)y = 2x$, with $y(0) = a$, $y'(0) = b$.

16. $3y'' + 2xy' + (1-2x^2)y = 1 + 2x$, with $y(0) = a$, $y'(0) = b$.

8.2 A General Approach to Power Series Solutions of Homogeneous Equations

The method developed in Section 8.1 works satisfactorily if only the first few terms in a power series solution are required, but it has the disadvantage that a separate calculation is required each time a coefficient is determined. The present section shows how in many cases this difficulty can be overcome by introducing a systematic and simple way of generating arbitrarily many terms in a power series solution of the homogeneous linear differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0 \tag{4}$$

about a point x_0, when $a(x)$, $b(x)$, and $c(x)$ are polynomials with $a(x_0) \neq 0$.

The approach enables the coefficients of the power series solution to be determined by means of a *recurrence relation* that relates a few consecutive coefficients in the series. This has the advantage that once the first few coefficients in the series expansion have been found, the rest can be generated by means of the recurrence relation.

There will be no loss of generality if the approach is based on an expansion about the origin, because if one is required about an arbitrary point $x = x_0$, the change of variable $X = x - x_0$ will shift the point $x = x_0$ to $X = 0$. For example, suppose a solution of

$$y'' + (2+3x)y' + x^2 y = 0$$

is required about the point $x = 1$, corresponding to the specification of the initial conditions for $y(1)$ and $y'(1)$ at $x = 1$. Setting $X = x - 1$ and $y(x) = Y(x-1) = Y(X)$, it follows that $y(1) = Y(0)$, $dy/dx = dY/dX$, $d^2y/dx^2 = d^2Y/dX^2$, and $x = X + 1$, so in terms of the new variables X and Y the equation and initial conditions become

$$Y'' + (5+3X)Y' + (1+X)^2 Y = 0, \quad \text{with } Y(0) = y(1),\ Y'(0) = y'(1).$$

Setting $X = x - 1$ in the power series solution of this equation expanded about $X = 0$ reduces it to the solution of the original equation expanded about $x = 1$.

The approach we now describe involves seeking a solution in the form of a general power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{5}$$

and finding a relationship between the coefficients a_n by substituting (5) into the

homogeneous differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0. \tag{6}$$

We will assume that the coefficients $a(x)$, $b(x)$, and $c(x)$ in the differential equation are polynomials in x, and so are analytic at $x = 0$, and also that $a(0) \neq 0$. If (5) is to be a solution of (6), it must satisfy the differential equation for all x, but this will only be possible if, after combining terms, the coefficient of each power of x in the new power series is zero. It will be seen later that it is this last requirement that leads to the determination of the coefficients a_n in terms of a recurrence relation.

Before illustrating the approach by means of an example, we first find expressions for the derivatives $y'(x)$ and $y''(x)$ that will be needed in the calculation. Writing out the first few terms of $y(x)$ in (5) gives

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = \sum_{n=0}^{\infty} a_nx^n. \tag{7}$$

Differentiating this expression term by term with respect to x, which is permitted for x inside the interval of convergence of the series, we arrive at the result

$$y'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots = \sum_{n=1}^{\infty} na_nx^{n-1}, \tag{8}$$

and after a further differentiation we have

$$y''(x) = 2a_2 + 2\cdot 3a_3x + 3\cdot 4a_4x^2 + \cdots = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}. \tag{9}$$

In what is to follow it will be important to remember that the summation in (8) starts at $n = 1$, whereas the summation in (9) starts at $n = 2$.

EXAMPLE 8.4 Find the recurrence relation that must be satisfied by coefficients in the series solution of the differential equation

$$y'' + 2xy' + (1 + x^2)y = 0$$

when the expansion is about the origin. Solve the initial value problem for this differential equation given that $y(0) = 3$ and $y'(0) = -1$.

Solution Substituting $y(x) = \sum_{n=0}^{\infty} a_nx^n$ into the differential equation and using (8) and (9) gives

$$\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + 2x\sum_{n=1}^{\infty} na_nx^{n-1} + (1 + x^2)\sum_{n=0}^{\infty} a_nx^n = 0.$$

Taking the factor $2x$ in the second term and the factor x^2 in the third term under their respective summation signs allows the equation to be written in the form

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0.$$

The powers of x in the first and last summations are different from those in the middle two summations, so before combining the summations in order to find the coefficient of each power of x, it will first be necessary to change the power of x in the first and last terms from $n-2$ and $n+2$ to n.

In the first summation we set $m = n-2$, causing the summation to become

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m.$$

However, m is simply a summation index that can be replaced by any other symbol, so we will replace it by n to obtain the equivalent expression

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n.$$

Similarly, by setting $m = n+2$ in the last summation, and then replacing m by n, we find that

$$\sum_{n=0}^{\infty} a_n x^{n+2} \quad \text{becomes} \quad \sum_{n=2}^{\infty} a_{n-2} x^n.$$

We now substitute these last two results into the series solution of the differential equation to obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0,$$

where now each summation involves x^n, though not all summations start from $n = 0$.

Separating out the terms corresponding to $n = 0$ and $n = 1$, and collecting all the remaining terms under a single summation sign in which the summation starts from $n = 2$, this becomes

$$2a_2 + a_0 + (6a_3 + 3a_1)x + \sum_{n=2}^{\infty} [(n+2)(n+1)a_{n+2} + 3a_n + a_{n-2}]x^n = 0.$$

As already remarked, if this power series is to be a solution of the differential equation it must satisfy the equation identically for all x, but this will only be possible if in the foregoing expression the coefficient of each power of x vanishes. Applying this condition to the preceding series we find that for it to vanish identically for all x,

deriving and using a recurrence relation

$$\begin{aligned} &(\text{coefficient of } x^0) \quad 2a_2 + a_0 = 0 \\ &(\text{coefficient of } x) \quad 6a_3 + 3a_1 = 0 \end{aligned}$$

and

$$(\text{coefficient of } x^n) \quad (n+2)(n+1)a_{n+2} + 3a_n + a_{n-2} = 0, \quad \text{for } n \geq 2.$$

The first condition shows that

$$a_2 = -\frac{1}{2}a_0,$$

while the second condition shows that

$$a_3 = -\frac{1}{2}a_1,$$

where a_0 and a_1 are arbitrary constants.

The third condition is a **recurrence relation** (also called a **recursion relation** or an **algorithm**) that in this case relates three coefficients whose indices differ by 2, so given a_{n-2} and a_n we can find a_{n+2} for $n = 2, 3, 4, \ldots$.

We now show how to determine the first few coefficients a_n by writing the recursion relation in the form

$$a_{n+2} = -\frac{[(2n+1)a_n + a_{n-2}]}{(n+1)(n+2)}$$

and setting $n = 2, 3, 4, \ldots$.

For $n = 2$, after using $a_2 = -\frac{1}{2}a_0$, we find that

$$a_4 = -\frac{(5a_2 + a_0)}{12} = \frac{a_0}{8},$$

whereas for $n = 3$, after using $a_3 = -\frac{1}{2}a_1$, we find that

$$a_5 = -\frac{(7a_3 + a_1)}{20} = \frac{a_1}{8}.$$

Continuing this process generates the coefficients

$$a_6 = -\frac{a_0}{48}, \quad a_7 = -\frac{a_1}{48}, \quad a_8 = \frac{a_0}{384}, \quad a_9 = \frac{a_1}{384}, \ldots.$$

Thus, all the coefficients with even suffixes are determined in terms of the arbitrary constant a_0, whereas all the coefficients with odd suffixes are determined in terms of the arbitrary constant a_1.

Substituting these coefficients into the power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and grouping terms gives

$$y(x) = a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \frac{1}{384}x^8 - \cdots\right)$$
$$+ a_1\left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5 - \frac{1}{48}x^7 + \frac{1}{384}x^9 - \cdots\right).$$

As the coefficients a_0 and a_1 are arbitrary, the functions represented by the series

$$y_1(x) = 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \frac{1}{384}x^8 - \cdots$$

and

$$y_2(x) = x - \frac{1}{2}x^3 + \frac{1}{8}x^5 - \frac{1}{48}x^7 + \frac{1}{384}x^9 - \cdots$$

are seen to be the two linearly independent solutions known to be associated with a homogeneous linear second order equation. So all possible solutions of the differential equation can be written in the form

$$y(x) = C_1 y_1(x) + C_2 y_2(x),$$

with C_1 and C_2 arbitrary constants, where to reconcile this result with our previous notation we notice that C_1 and C_2 have been written in place of a_0 and a_1.

To solve the initial value problem the constants C_1 and C_2 must be chosen such that $y(0) = 3$ and $y'(0) = -1$, so

$$3 = C_1 y_1(0) + C_2 y_2(0) \quad \text{and} \quad -1 = C_1 y_1'(0) + C_2 y_2'(0),$$

but $y_1(0) = 1$, $y_2(0) = 0$, and differentiation of the expressions for $y_1(x)$ and $y_2(x)$ shows that $y_1'(0) = 0$ and $y_2'(0) = 1$, so solving for C_1 and C_2 gives $C_1 = 3$ and $C_2 = -1$, showing that the required solution to the initial value problem is

$$y(x) = 3y_1(x) - y_2(x).$$

■

The coefficients of the power series expansions for $y_1(x)$ and $y_2(x)$ in the last example were sufficiently complicated that no attempt was made to deduce their general forms and they were merely generated from the recurrence relation. The next example is simpler, and we use it to illustrate the type of argument that is necessary when attempting to arrive at the form of the general term in a power series solution of a homogeneous linear differential equation. There are no specific rules to follow when seeking the form of a general term in a series, and success depends on experience and the ability to recognize the pattern of signs and numbers forming the coefficients.

EXAMPLE 8.5 Find two linearly independent solutions of

$$y'' + xy' + y = 0,$$

when the series expansion is about the origin, and hence solve the initial value problem for which $y(0) = 1$ and $y'(0) = 0$.

Solution Substituting results (7) to (9) into the differential equation gives

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Shifting the summation index in the first term, taking the factor x under the second summation and separating out the constant term, as in Example 8.4, gives

$$2a_2 + a_0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} + (n+1)a_n]x^n = 0.$$

Equating the coefficient of each power of x to zero, as in Example 8.4, shows that

$$2a_2 + a_0 = 0, \quad \text{so } a_2 = -\frac{a_0}{2},$$

and

$$(n+2)(n+1)a_{n+2} + (n+1)a_n = 0 \quad \text{for } n \geq 1,$$

but as $n + 1 \neq 0$ this last condition reduces to the simpler *recurrence relation*

$$a_{n+2} = -\frac{a_n}{n+2}, \quad \text{for } n = 1, 2, \ldots.$$

It follows directly from the recurrence relation that all even coefficients are multiples of a_0 and all odd coefficients are multiples of a_1 with

$$a_3 = -\frac{a_1}{3}, \quad a_4 = -\frac{a_2}{4} = \frac{a_0}{2\cdot 4}, \quad a_5 = -\frac{a_3}{5} = \frac{a_1}{3\cdot 5}, \quad a_6 = -\frac{a_4}{6} = -\frac{a_0}{2\cdot 4\cdot 6},$$

$$a_7 = -\frac{a_5}{7} = -\frac{a_1}{3\cdot 5\cdot 7}, \quad a_8 = -\frac{a_6}{8} = \frac{a_0}{2\cdot 4\cdot 6\cdot 8}, \quad a_9 = -\frac{a_7}{9} = \frac{a_1}{3\cdot 5\cdot 7\cdot 9}, \dots,$$

where a_0 and a_1 are arbitrary constants.

It is apparent that the pattern of coefficients with even suffixes differs from the one for coefficients with odd suffixes, so each must be considered separately. Starting with the coefficients with even suffixes, we use the fact that if $m = 1, 2, \dots,$ then $2m$ is an even number. A little experimentation shows that the signs of the terms with even suffixes are given by the factor $(-1)^m$.

Noticing that $a_2, a_4, a_6,$ and a_8 can be written in the form

$$a_2 = \frac{(-1)a_0}{2}, \quad a_4 = \frac{1}{2\cdot 4}\frac{(-1)^2\,a_0}{2^2 2!}, \quad a_6 = \frac{-a_0}{2\cdot 4\cdot 6} = \frac{(-1)^3 a_0}{2^3 3!},$$

$$a_8 = \frac{(-1)^4 a_0}{2\cdot 4\cdot 6\cdot 8} = \frac{(-1)^4 a_0}{2^4 4!}$$

suggests that if we set $n = 2m$, for $m = 0, 1, 2, \dots,$ the even numbered terms can be written

$$a_{2m} = \frac{(-1)^m}{2^m m!}a_0.$$

A formal proof that this is the general coefficient in the series involving even powers of x can be obtained by mathematical induction, but we leave this as an exercise.

It is now necessary to consider the coefficients with odd suffixes, and to do this we use the fact that if $m = 1, 2, 3, \dots,$ then $2m + 1$ is an odd number. Noticing that the coefficients $a_3, a_5, a_7,$ and a_9 can be written

$$a_3 = \frac{-a_1}{3} = \frac{(-1)2a_1}{3!}, \quad a_5 = \frac{a_1}{3\cdot 5} = \frac{(-1)^2\,2\cdot 4a_1}{1\cdot 2\cdot 3\cdot 4\cdot 5} = \frac{(-1)^2\,2^2 2!}{5!},$$

$$a_7 = \frac{-a_1}{3\cdot 5\cdot 7} = \frac{(-1)^3\,2\cdot 4\cdot 6a_1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} = \frac{(-1)^3\,2^3 3! a_1}{7!},$$

$$a_9 = \frac{a_1}{3\cdot 5\cdot 7\cdot 9} = \frac{(-1)^4\,2\cdot 4\cdot 6\cdot 8a_1}{9!} = \frac{(-1)^4\,2^4 4! a_1}{9!}$$

suggests that the coefficients in the series of odd powers of x can be written

$$a_{2m+1} = \frac{(-1)^m\,2^m m!}{(2m+1)!}a_1.$$

Here again we leave as an exercise the task of giving an inductive proof that this is, indeed, the coefficient of the general term in the series involving odd powers of x.

The solution of the differential equation has now separated into two series, one multiplied by a_0 containing only even powers of x and the other multiplied by a_1 containing only odd powers of x, so the solution becomes

$$y(x) = a_0\sum_{m=0}^{\infty}\frac{(-1)^m\,x^{2m}}{2^m m!} + a_1\sum_{m=0}^{\infty}\frac{(-1)^m\,2^m m! x^{2m+1}}{(2m+1)!}.$$

As a_0 and a_1 are arbitrary constants, and the two series are not proportional, it follows that two linearly independent solutions of the differential equation are

$$y_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^m m!} \quad \text{and} \quad y_2(x) = \sum_{m=0}^{\infty} \frac{(-1)^m 2^m m! x^{2m+1}}{(2m+1)!},$$

so the general solution is

$$y(x) = C_1 y_1(x) + C_2 y_2(x),$$

where C_1 and C_2 are arbitrary constants.

Using the series for $y_1(x)$ and $y_2(x)$, simple calculation gives $y_1(0) = 1$, $y_1'(0) = 0$, $y_2(0) = 0$, and $y_2'(0) = 1$, so the initial conditions $y(0) = 1$, $y'(0) = 0$ will be satisfied if the constants C_1 and C_2 are such that

$$1 = C_1 y_1(0) + C_2 y_2(0) \quad \text{and} \quad 0 = C_1 y_1'(0) + C_2 y_2'(0).$$

This pair of equations has the solution $C_1 = 1$ and $C_2 = 0$, so the solution of the initial value problem becomes

$$y(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^m m!}.$$

Rewriting this as

$$y(x) = \sum_{m=0}^{\infty} \frac{(-x^2/2)^m}{m!},$$

we recognize that the solution is simply $y(x) = \exp(-x^2/2)$, so this series is known to converge for all x.

Finally, to complete our examination of the two linearly independent solutions, let us find the radius of convergence of the second solution $y_2(x)$. The formula for the radius of convergence R based on the ratio test requires all powers of x to be present, whereas the series $y_2(x)$ only contains odd powers of x, so we must modify the series before using the test. All that is necessary is to set $z = x^2$ and to write the series in the form

$$y_2(x) = x \sum_{m=1}^{\infty} \frac{(-1)^m 2^m m!}{(2m+1)!} z^m,$$

for now the radius of convergence of the series in z can be found. The coefficient a_m of z^m is

$$a_m = (-1)^m \frac{2^m m!}{(2m+1)!},$$

so the radius of convergence R is given by

$$\begin{aligned} R &= \lim_{m\to\infty} 1/|a_{m+1}/a_m| = \lim_{m\to\infty} \left| \frac{a_m}{a_{m+1}} \right| = \lim_{m\to\infty} \left(\frac{2^m m!}{(2m+1)!} \frac{(2m+3)!}{2^{m+1}(m+1)!} \right) \\ &= \lim_{m\to\infty} (2m+3) = \infty. \end{aligned}$$

As the series in z has an infinite radius of convergence, so also does the original series involving odd powers of x. This means that the general solution

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

is valid for all real x. ■

Legendre's equation

An important application of the power series method of solution is to the **Legendre differential equation**

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \tag{10}$$

in which $\alpha \geq 0$ is a real parameter. The equation arises in a variety of applications, but mainly in connection with physical problems in which spherical symmetry is present. It will be seen later that the equation finds its origin in the study of Laplace's equation when expressed in spherical coordinates. Solutions of (10) are called **Legendre functions**, and they are examples of **special functions**, or so-called **higher transcendental functions**, as distinct from elementary functions such as sine, cosine, exponential, and logarithm. We first develop the series solutions for arbitrary $\alpha \geq 0$, and then consider the cases $\alpha = n = 0, 1, 2, \ldots,$ which lead to a special class of polynomial solutions $P_n(x)$ called **Legendre polynomials** in which n is the degree of the polynomial. The important properties of Legendre polynomials will be examined later when the topic of orthogonal functions is introduced.

The coefficients of Legendre's equation are all analytic at the origin and the leading coefficient $(1 - x^2)$ only vanishes at $x = \pm 1$, so a power series solution can be expected to exist in the interval $-1 < x < 1$. Substituting (7) to (9) in (10) leads to the equation

$$(1 - x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} na_n x^{n-1} + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n x^n = 0.$$

Proceeding as in Example 8.4, this can be rewritten as

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 2na_n x^n + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n x^n = 0,$$

so equating each coefficient to zero in the usual manner gives the following:
Coefficient of x^0:

$$2a_2 + \alpha(\alpha+1)a_0 = 0,$$

Coefficient of x:

$$6a_3 - 2a_1 + \alpha(\alpha+1)a_1 = 0,$$

Coefficient of x^n for $n \geq 2$:

$$(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + \alpha(\alpha+1)a_n = 0.$$

Solving the first two equations gives

$$a_2 = -\frac{\alpha(\alpha+1)}{2}a_0 \quad \text{and} \quad a_3 = \frac{[2 - \alpha(\alpha+1)]}{6}a_1,$$

whereas the third result gives the recurrence relation

$$a_{n+2} = -\frac{(\alpha - n)(\alpha + n + 1)}{(n+2)(n+1)}a_n \quad \text{for } n \geq 2. \tag{11}$$

Straightforward calculations show that the first few coefficients are given by

$$a_2 = -\frac{\alpha(\alpha+1)}{2!}a_0, \quad a_3 = -\frac{(\alpha-1)(\alpha+2)}{3!}a_1,$$
$$a_4 = \frac{(\alpha-2)\alpha(\alpha+1)(\alpha+3)}{4!}a_0, \quad a_5 = \frac{(\alpha-3)(\alpha-1)(\alpha+2)(\alpha+4)}{5!}a_1,$$
$$a_6 = -\frac{(\alpha-4)(\alpha-2)\alpha(\alpha+1)(\alpha+3)(\alpha+5)}{6!}a_0.$$

Thus, the coefficients of the even powers of x are all multiples of a_0, whereas the coefficients of the odd powers of x are all multiples of a_1, where a_0 and a_1 are arbitrary real numbers. Substituting these coefficients into the series

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = \sum_{n=0}^{\infty} a_nx^n$$

shows that the general solution of the Legendre differential equation can be written

$$y(x) = a_0y_1(x) + a_1y_2(x), \tag{12}$$

where

$$y_1(x) = 1 - \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{(\alpha-2)\alpha(\alpha+1)(\alpha+3)}{4!}x^4 - \cdots, \tag{13}$$

and

$$y_2(x) = x - \frac{(\alpha-1)(\alpha+2)}{3!}x^3 + \frac{(\alpha-3)(\alpha-1)(\alpha+2)(\alpha+4)}{5!}x^5 - \cdots. \tag{14}$$

As the solutions $y_1(x)$ and $y_2(x)$ are not proportional, they must be linearly independent solutions of the Legendre equation (10). We leave as an exercise the task of showing that each series is convergent in the interval $-1 < x < 1$, so the general solution (12) has this same interval of convergence.

Examination of the recurrence relation (11) shows that if $\alpha = n$ is a nonnegative integer, the terms $a_{n+2} = a_{n+4} = a_{n+6} = \cdots$ all vanish. Thus, if $\alpha = n$ is even, the series $y_1(x)$ will reduce to a polynomial of degree n in even powers of x, whereas if $\alpha = n$ is odd the series $y_2(x)$ will reduce to a polynomial of degree n in odd powers of x.

The solution $y(x)$ reduces to the following polynomials when $n = 0, 1, 2, 3, 4$:

Case $n = 0$:

$$y(x) = a_0,$$

Case $n = 1$:

$$y(x) = a_1x,$$

Case $n = 2$:

$$y(x) = a_0(1 - 3x^2),$$

Case $n = 3$:

$$y(x) = a_1\left(x - \frac{5}{3}x^3\right),$$

Case $n = 4$:

$$y(x) = a_0\left(1 - 10x^2 + \frac{35}{3}x^4\right).$$

When α is a nonnegative integer, after suitable scaling the foregoing polynomials are denoted by $P_n(x)$ and called **Legendre polynomials of degree *n***. The standard scaling adopted involves choosing the arbitrary multiplier of each polynomial such that $P_n(1) = 1$ for $n = 0, 1, 2, \ldots$. When this is done the first few Legendre polynomials become

Legendre polynomials

Even polynomials	*Odd polynomials*
$P_0(x) = 1$	$P_1(x) = x$
$P_2(x) = \frac{1}{2}(3x^2 - 1)$	$P_3(x) = \frac{1}{2}(5x^3 - 3x)$
$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$	$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

A general expression for $P_n(x)$ can be obtained by writing the recurrence relation (11) in the form

$$a_r = \frac{(r+2)(r+1)}{(r-n)(n+r+1)}a_{r+2} \quad \text{for } r \le n-2$$

and finding that

$$a_n = \frac{1 \cdot 3 \cdot 4 \cdots (2n-1)}{n!} = \frac{(2n)!}{2^n(n!)^2} \quad \text{for } n = 1, 2, 3, \ldots,$$

in order to make $P_n(1) = 1$. As a result, the following expressions for $P_n(x)$ are obtained.

For *even* polynomials:

$$P_{2n}(x) = \sum_{r=0}^{n}(-1)^r \frac{(4n-2r)!}{2^{2n}r!(2n-r)!(2n-2r)!}x^{2n-2r}, \quad n = 0, 1, 2, \ldots. \tag{15a}$$

For *odd* polynomials:

$$P_{2n+1}(x) = \sum_{r=0}^{n}(-1)^r \frac{(4n-2r+2)!}{2^{2n+1}r!(2n-r+1)!(2n-2r+1)!}x^{2n-2r+1}, \quad n = 0, 1, 2, \ldots. \tag{15b}$$

Two alternative definitions of Legendre polynomials are to be found in Exercises 16 and 18 at the end of this section.

Results (15a, b) provide a general definition for a Legendre polynomial of any order, though when only a few low order polynomials are required it is often more convenient to generate them by means of the following recurrence relation that

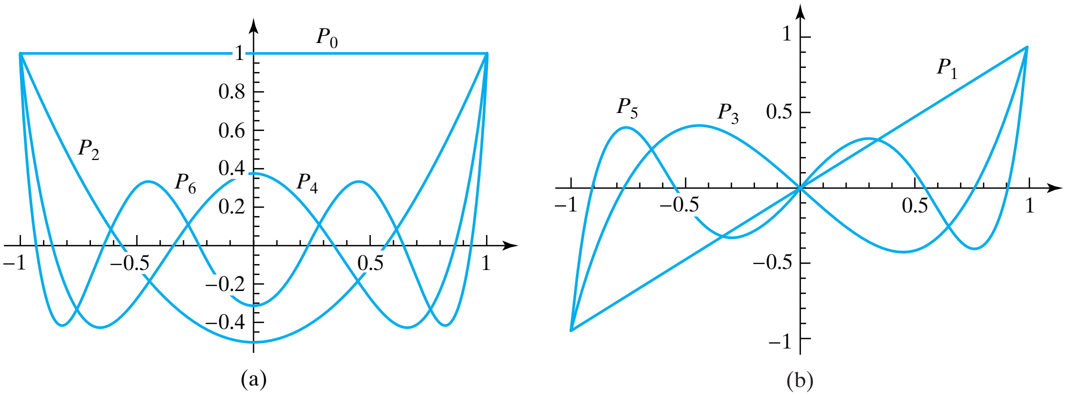


FIGURE 8.1 (a) Even Legendre polynomials. (b) Odd Legendre polynomials.

determines $P_{n+1}(x)$ in terms of $P_n(x)$ and $P_{n-1}(x)$:

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0, \tag{16}$$

recurrence relation for Legendre polynomials

for $n = 1, 2, 3, \ldots$. A derivation of this recurrence relation is to be found in Exercise 17 at the end of this section.

As an example of the use of (16) we set $n = 2$ to obtain

$$P_3(x) = \frac{1}{3}[5xP_2(x) - 2P_1(x)],$$

but $P_1(x) = x$ and $P_2(x) = \dfrac{1}{2}(3x^2 - 1)$, so substituting these expressions, we find $P_3(x) = \frac{1}{2}(5x^3 - 3x)$.

Graphs of the first few Legendre polynomials $P_n(x)$ are given in Fig. 8.1.

ADRIEN-MARIE LEGENDRE (1752–1833)
A French mathematician educated at a college in Paris whose remarkable mathematical ability enabled him to be appointed to the position of professor of mathematics at a military school in Paris. His work on the motion of projectiles in a resisting medium won him a prize offered by the Royal Academy in Berlin. He was subsequently appointed professor at the Normal School in Paris and his contributions as an analyst were second only to those of Laplace and Lagrange, who were his contemporaries. In addition to his contributions to the development of the calculus, he made major contributions to the study of elliptic functions.

For more information about Legendre polynomials, and for applications to boundary value problems, see Chapters 5 and 8 of reference [3.7]. Recurrence relations satisfied by Legendre polynomials and other orthogonal polynomials are to be found in Chapter 22 of reference [G.1], and also in Chapter 18 of reference [G.3].

Another important and useful differential equation with a power series solution is the **Chebyshev equation**,

Chebyshev equation

$$(1 - x^2)y'' - xy' + \alpha y = 0. \tag{17}$$

The coefficients are all analytic functions and the leading coefficient $(1 - x^2)$ only vanishes at $x = \pm 1$, so a power series solution can be found in the interval

$-1 \le x \le 1$. Proceeding as with Legendre's equation we find

$$(1 - x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} na_n x^{n-1} + \alpha \sum_{n=0}^{\infty} a_n x^n = 0,$$

or after a shift of summation index,

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} na_n x^n + \alpha \sum_{n=0}^{\infty} a_n x^n = 0.$$

If we combine summations, this becomes

$$(1 \cdot 2a_2 + \alpha a_0) + [2 \cdot 3a_3 + (\alpha - 1)a_1]x + \sum_{n=2}^{\infty}[(n+1)(n+2)a_{n+2} + (\alpha - n^2)a_n]x^n = 0.$$

Equating the coefficients of each power of x to zero gives

$$a_2 = -\frac{\alpha}{2!}a_0, \quad a_3 = \frac{(1-\alpha)}{3!}a_1,$$

and the recurrence relation

$$a_{n+2} = \frac{(n^2 - \alpha)}{(n+1)(n+2)}a_n, \quad n = 2, 3 \ldots .$$

Thus,

$$a_4 = \frac{(2^2 - \alpha)}{3 \cdot 4}a_2 = -\frac{\alpha(2^2 - \alpha)}{4!}a_0$$

$$a_5 = \frac{(3^2 - \alpha)}{4 \cdot 5}a_3 = \frac{(1-\alpha)(3^2 - \alpha)}{5!}a_1$$

$$\ldots .$$

Using these coefficients in the original power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$ gives the solution of the Chebyshev equation in the form

$$y(x) = a_0 y_0(x) + a_1 y_1(x),$$

where

$$y_0(x) = a_0\left[1 - \frac{\alpha}{2!}x^2 - \frac{\alpha(2^2 - \alpha)}{4!}x^4 - \frac{\alpha(2^2 - \alpha)(4^2 - \alpha)}{6!}x^6 - \cdots\right]$$

and

$$y_1(x) = a_1\left[x + \frac{(1-\alpha)}{3!}x^3 + \frac{(1-\alpha)(3^2 - \alpha)}{5!}x^5 + \cdots\right].$$

In applications of this equation to approximation theory, numerical analysis, and elsewhere, it is usual that $\alpha = m^2$, where $m = 0, 1, 2, \ldots$. Inspection of $y_0(x)$ shows that when m is even, the solution reduces to a polynomial of degree m in even powers of x, whereas when m is odd $y_1(x)$ reduces to a polynomial of degree m in odd powers of x.

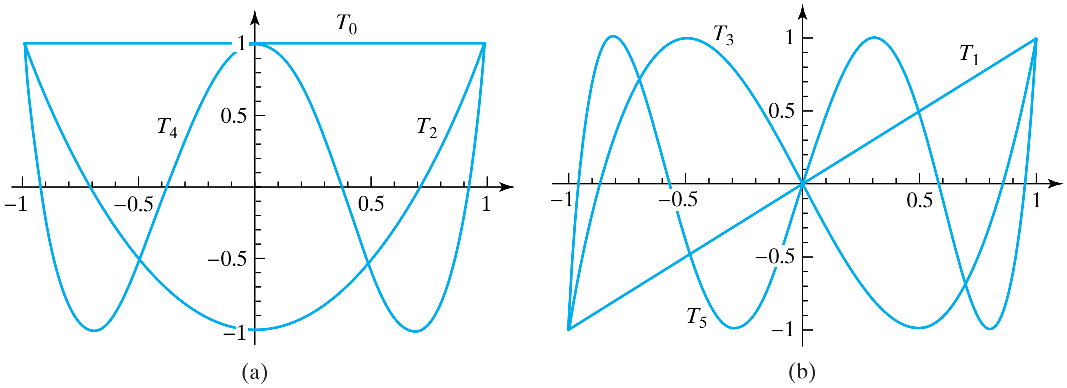


FIGURE 8.2 (a) Even Chebyshev polynomials. (b) Odd Chebyshev polynomials.

Chebyshev polynomials

As the polynomials are solutions of a homogeneous differential equation, the scale factors for each polynomial can be chosen arbitrarily, so by convention they are chosen such that the term with the largest power of x is positive and the polynomial is free from fractional coefficients. These polynomials are called **Chebyshev polynomials**, and they are denoted by $T_n(x)$. The first six Chebyshev polynomials are:

Even polynomials	*Odd polynomials*
$T_0(x) = 1$	$T_1(x) = x$
$T_2(x) = 2x^2 - 1$	$T_3(x) = 4x^3 - 3x$
$T_4(x) = 8x^4 - 8x^2 + 1$	$T_5(x) = 16x^5 - 20x^3 + 5x$

recurrence relation for Chebyshev polynomials

Using the forms for $T_{n+1}(x)$, $T_n(x)$ and $T_{n-1}(x)$ obtained from $y_0(x)$ and $y_1(x)$, it can be shown that Chebyshev polynomials obey the following recurrence relation:

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0. \tag{18}$$

When used with the polynomials just listed, this recurrence relation is the simplest way of generating higher order polynomials. Graphs of the first six Chebyshev polynomials are shown in Fig. 8.2.

For applications of Chebyshev polynomials to numerical analysis see, for example, references [8.3] to [8.5].

PAFNUTI LIWOWICH CHEBYSHEV (1821–1894)
A distinguished Russian mathematician who was professor of mathematics at the University of Petrograd (now St. Petersburg). He made many contributions to analysis and number theory. There are many variations of the transliteration of his name, the most common probably being Tchebycheff.

Summary

This section showed how to find a series solution, expanded about the origin, of a homogeneous linear second order variable coefficient differential equation with polynomial coefficients, when the solution can be obtained in the form of a general power series with unknown coefficients. By substituting this series into the differential equation, grouping corresponding powers of x, and requiring the coefficient of each power of x to vanish

identically, a recurrence relation connecting the unknown coefficients was obtained and used to find the coefficients of the power series in terms of two arbitrary constants a_0 and a_1. The general solution was seen to be the sum of two linearly independent power series with known coefficients, one multiplied by a_0 and the other by a_1. Two important special cases were considered that gave rise to polynomial solutions of the important and useful Legendre and Chebyshev equations.

EXERCISES 8.2

Find the first six terms in the power series expansion of each of the following initial value problems.

1. $y'' + (x - x^2)y' + y = 0$, with $y(0) = 2$, $y'(0) = -3$.

2. $2y'' + xy' + 2(1 + x)y = 0$, with $y(0) = -2$, $y'(0) = 1$.

3. $y'' + (1 + x^2)y' + xy = 0$, with $y(0) = 1$, $y'(0) = -3$.

4. $y'' - 3xy' + 2y = 0$, with $y(0) = 1$, $y'(0) = 1$.

5. $(1 - x^2)y'' + xy' - y = 0$, with $y(0) = 2$, $y'(0) = -1$.

6. $y'' + x^2y' + 2xy = 0$, with $y(0) = 3$, $y'(0) = -2$.

7. $y'' + 2(1 - x)y' - 3xy = 0$, with $y(0) = 1$, $y'(0) = -1$.

8. $(1 - x)y'' + 2xy' + (1 + x)y = 0$, with $y(0) = 4$, $y'(0) = -2$.

9. $(1 - 2x^2)y'' + 2y' + 3y = 0$, with $y(0) = 1$, $y'(0) = -1$.

10. $(1 + 2x^2)y'' + 3xy' + y = 0$, with $y(0) = 2$, $y'(0) = -2$.

11. $(2x^2 - 1)y'' + (1 + x)y' + 2y = 0$, with $y(0) = 1$, $y'(0) = 4$.

12. $y'' + (1 + 2x)y' + xy = 0$, with $y(2) = 1$, $y'(2) = 0$.

13. $(2 + x)y'' + 3(1 + x)y' + 2y = 0$, with $y(1) = 2$, $y'(1) = -3$.

14. $(x^2 - 2x + 2)y'' + (x - 1)y' - 3y = 0$, with $y(-1) = 1$, $y'(-1) = 2$.

15. $(1 - x)y'' + 2xy' - 2xy = 0$, with $y(2) = 1$, $y'(2) = 5$.

16. An alternative definition of the Legendre polynomial $P_n(x)$ is provided by the formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n,$$

called the **Rodrigues formula**. Use the formula to compute $P_4(x)$ and $P_5(x)$.

17.* Set $u = (x^2 - 1)^n$ and use repeated differentiation of the Rodrigues formula to verify that $P_n(x)$ is a Legendre polynomial by showing it satisfies the Legendre differential equation

$$(1 - x^2)P_n''(x) - 2xP_n'(x) + n(n + 1)P_n(x) = 0.$$

18.* The function

$$G(x, t) = (1 - 2xt + t^2)^{-1/2}$$

is called the **generating function** for Legendre polynomials. It has the property that when expanded as a power series in t the coefficient of t^n is $P_n(x)$, so that

$$G(x, t) = P_0(x) + P_1(x)t + P_2(x)t^2 + \cdots.$$

Set $u = -2xt + t^2$ and expand $(1 + u)^{-1/2}$ by the binomial theorem. Collect all the terms in x multiplying t^5 and hence verify that the coefficient of t^5 is $P_5(x)$.

19.* Show that the generating function defined in Problem 18 satisfies the differential equation

$$(1 - 2xt + t^2)\frac{\partial G}{\partial t} - (x - t)G = 0$$

for arbitrary t. As the result must be an identity in t, the consequence of substituting

$$G(x, t) = P_0(x) + P_1(x)t + P_2(x)t^2 + \cdots$$

into the differential equation must be such that terms in x multiplying each power of t vanish. Collect the terms multiplying t^n, and hence establish the Legendre polynomial recurrence relation

$$(n + 1)P_{n+1}(x) - (2n + 1)xP_n(x) + nP_{n-1}(x) = 0$$

for $n = 1, 2, \ldots$. This result is called the **Bonnet recurrence relation**.

20.* The **electrostatic potential** ϕ at a point in a vacuum distant d from a charge Q is given by $\phi = Q/d$. Use the Legendre polynomial generating function

$$G(r, t) = \frac{1}{(1 - 2rt + t^2)^{1/2}},$$

together with the result from elementary trigonometry

$$r = \left(r_1^2 + r_2^2 - 2r_1r_2\cos\theta\right)^{1/2},$$

to show that the electrostatic potential at point A due to a charge Q at B in Fig. 8.3 is given by

$$\frac{Q}{r} = \frac{1}{r_2}\left[P_0(\cos\theta) + \left(\frac{r_1}{r_2}\right)P_1(\cos\theta) + \left(\frac{r_1}{r_2}\right)^2 P_2(\cos\theta) + \cdots\right], \text{ for } r_1/r_2 < 1.$$

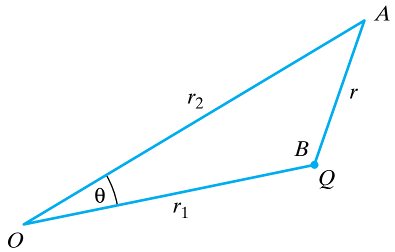


FIGURE 8.3 A point charge Q at B distant r from A.

8.3 Singular Points of Linear Differential Equations

In Section 8.2 the power series method was used to find a solution of a homogeneous variable coefficient differential equation of the form

$$a(x)y'' + b(x)y' + c(x)y = 0. \tag{19}$$

It was seen that the method could be applied about any point x_0 at which the coefficients of the differential equation are analytic and $a(x_0) \neq 0$. Expressed differently, when (19) is written in the standard form

$$y'' + P(x)y' + Q(x)y = 0, \tag{20}$$

with

$$P(x) = \frac{b(x)}{a(x)} \quad \text{and} \quad Q(x) = \frac{c(x)}{a(x)}, \tag{21}$$

the power series method can be applied to develop a solution about any point x_0 at which the functions $P(x)$ and $Q(x)$ are analytic.

regular and singular points

Points where $P(x)$ and $Q(x)$ are analytic are called **regular points** of the differential equation, and points where at least one is not analytic are called **singular points**.

Equation (20) will be said to have a **regular singular point** at x_0 if the functions

$$(x - x_0)P(x) \quad \text{and} \quad (x - x_0)^2 Q(x)$$

are analytic at x_0, and so have Taylor series expansions about x_0. If at least one of these functions is not analytic at x_0, the point will be said to be an **irregular singular point**.

EXAMPLE 8.6

Identify the nature of the singular points of the following equations:

(a) $x^2y'' + xy' + (x^2 - n^2)y = 0$
(b) $(1 - x^2)y'' - 2xy' + n(n+1)y = 0, \quad (n = 0, 1, 2, \ldots)$

(c) $(1-x)y'' + 2(x-1)y' + xy = 0$

(d) $(x-1)^3 y'' + 3(x-1)^2 y' + y = 0$

Solution

(a) This is *Bessel's equation* of order n in which the functions $P(x) = 1/x$ and $Q(x) = (x^2 - n^2)/x^2$. Neither of these functions is analytic at the origin, so the origin is a singular point of Bessel's equation. However, as the functions $xP(x) = 1$ and $x^2 Q(x) = x^2 - n^2$ are both analytic at the origin, it follows that $x = 0$ is a regular singular point of Bessel's equation.

(b) This is *Legendre's equation* of order n in which $P(x) = -2x/(1-x^2)$ and $Q(x) = n(n+1)/(1-x^2)$. Neither of these functions is analytic at $x = \pm 1$, so these points are the singular points of the Legendre equation. Let us consider the singular point at $x = 1$. As the functions

$$(x-1)P(x) = 2x/(1+x) \quad \text{and} \quad (x-1)^2 Q(x) = n(n+1)(x-1)/(1+x)$$

are both analytic at $x = 1$, it follows that this is a regular singular point of Legendre's equation. A similar argument shows that $x = -1$ is also a regular singular point of the equation.

(c) In this case $P(x) = -2$ and $Q(x) = x/(1-x)$, and while $P(x)$ is analytic for all x the function $Q(x)$ is not analytic at $x = 1$, so this is a singular point of the equation. The functions $(x-1)P(x) = 2(1-x)$ and $(x-1)^2 Q(x) = x(1-x)$ are both analytic at $x = 1$, so $x = 1$ is a regular singular point of the equation.

(d) In this equation $P(x) = 3/(x-1)$ and $Q(x) = 1/(x-1)^3$ and neither function is analytic at $x = 1$, so this is a singular point of the equation. We have

$$(x-1)P(x) = 3 \quad \text{and} \quad (x-1)^2 Q(x) = \frac{1}{x-1},$$

and although the first of these functions is analytic for all x, the second is not analytic at $x = 1$, so $x = 1$ is an irregular singular point of the equation. ■

In the next section the power series method will be generalized to arrive at what is called the **Frobenius method**, which always generates two linearly independent solutions about a *regular singular point* of equation (20). As the behavior of solutions in a neighborhood of an irregular singular point can be shown to be very erratic, no further consideration will be given to solutions near such points.

shifting a singular point

Sometimes it is more convenient to consider an equation with a regular singular point located at the origin rather at some other point $x_0 \neq 0$. In such cases a singular point located at x_0 can always be shifted to the origin by making the change of variable $X = x - x_0$, as in Section 8.2.

EXAMPLE 8.7

Shift the singular point of the following equation to the origin:

$$(x-1)^2 y'' + 3(x+2)y' + 2y = 0.$$

Solution The equation has a regular singular point at $x = 1$, so we make the variable change $X = x - 1$ and set $y(x) = Y(x-1) = Y(X)$. The equation then becomes

$$X^2 Y'' + 3(X+3)Y' + 2Y = 0,$$

with a regular singular point now located at $X = 0$. ■

example showing why no power series solution exists about a singular point

To appreciate why an ordinary power series solution cannot be developed around a regular singular point, it will be sufficient to consider the Cauchy–Euler equation

$$x^2y'' + 3xy' + 2y = 0,$$

which has a regular singular point at the origin. This Cauchy–Euler equation was solved analytically in Example 6.10, where its solution was found to be

$$y(x) = C_1x^{-1}\cos(\ln|x|) + C_2x^{-1}\sin(\ln|x|).$$

The reason that no power series solution exists in this case is seen to be the presence of the factor x^{-1} and the function $\ln|x|$ in the analytical solution, neither of which can be expanded in a power series about the origin.

Summary

The regular and singular points of a general homogeneous second order linear variable coefficient differential equation were defined and illustrated by example. It was shown how, if necessary, a singular point occurring at $x = a$ could be shifted to the origin, and an example was used to demonstrate why an ordinary power series solution cannot be developed around a regular singular point.

EXERCISES 8.3

Identify the nature of the singular points in each of the following equations.

1. $(1 - x)^2y'' + 2(x - 1)y' + y = 0.$

2. $x^2y'' + 3x^2y' + (1 + x^2)y = 0.$

3. $(1 + x)^2y'' + 2y' + y = 0.$

4. $xy'' + (1 - x)y' + ny = 0 \quad (n > 0).$

5. $(x + 4)^3y'' + 2(x + 4)y' + xy = 0.$

6. $(x^2 - 4)y'' + (x + 3)y' - 5(x + 1)y = 0.$

7. $(3 - x)^2y'' + 4y' + \cos x(3 - x^2)y = 0.$

8. $x^2y'' + 8y' + 3xy = 0.$

8.4 The Frobenius Method

A generalization of the power series method that was introduced by Frobenius (1849–1917) enables a solution of a homogeneous linear differential equation to be developed about a regular singular point. He considered the differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0, \tag{22}$$

and established the following result that is stated without proof.

THEOREM 8.1

Frobenius theorem Let x_0 be a regular singular point of (22). Then, in some interval $0 < x - x_0 < d$, the equation will always possess at least one solution of the form

$$y(x) = (x - x_0)^c\left(a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots\right)$$

$$= (x - x_0)^c\sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

the Frobenius theorem and method of solution

where $a_0 \neq 0$ and c is a real or complex number. A second linearly independent solution of similar form will exist that may contain a logarithmic term, though with a different value of c and some other coefficients $b_0, b_1, b_2, \ldots$ in place of

the coefficients $a_0, a_1, a_2, \ldots$. Taken together, these two solutions form a basis of solutions for the differential equation. ■

GEORG FERDINAND FROBENIUS (1849–1917)
A German mathematician whose main research was in group theory and analysis. He worked in Zurich and Berlin and published his method for the series solution of linear ordinary differential equations in 1873.

For simplicity, and because of their frequent occurrence, in what follows we will develop the Frobenius method in terms of a slightly less general class of equations by setting $a(x) = x^2$ in (22). So we will consider the equation

$$x^2 y'' + b(x)y' + c(x)y = 0, \tag{23}$$

and write it in the standard form

$$y'' + P(x)y' + Q(x)y = 0, \tag{24}$$

where

$$P(x) = \frac{p(x)}{x} \quad \text{and} \quad Q(x) = \frac{q(x)}{x^2}, \tag{25}$$

and assume that $p(x)$ and $q(x)$ are analytic functions at $x = 0$. So we will only consider equations of the form (24) with regular singular points at the *origin*.

To determine the exponent c in Theorem 8.1 we substitute a solution of the form

$$y(x) = x^c \sum_{n=0}^{\infty} a_n x^n \tag{26}$$

into equation (24), where c is to be determined along with the coefficients a_n. When making this substitution we will need to use the following results obtained by differentiation of (26):

$$y'(x) = ca_0x^{c-1} + (c+1)a_1x^c + (c+2)a_2x^{c+1} + \cdots = \sum_{n=0}^{\infty}(n+c)a_nx^{n+c-1} \tag{27}$$

and

$$\begin{aligned} y''(x) &= c(c-1)a_0x^{c-2} + (c+1)ca_1x^{c-1} + (c+2)(c+1)a_2x^c + \cdots \\ &= \sum_{n=0}^{\infty}(n+c)(n+c-1)a_nx^{n+c-2}. \end{aligned} \tag{28}$$

As the functions $p(x)$ and $q(x)$ are assumed to be analytic at the origin, they can be expanded as the Maclaurin series

$$p(x) = p_0 + p_1x + p_2x^2 + \cdots \quad \text{and} \quad q(x) = q_0 + q_1x + q_2x^2 + \cdots. \tag{29}$$

Substituting (27) to (29) into (24) leads to the result

$$\begin{aligned} &x^{c-2}[c(c-1)a_0 + (c+1)ca_1x + \cdots] \\ &\quad + \left(p_0 + p_1x + p_2x^2 + \cdots\right)x^{c-2}(ca_0 + (c+1)a_1x + \cdots) \\ &\quad + x^{c-2}\left(q_0 + q_1x + q_2x^2 + \cdots\right)\left(a_0 + a_1x + a_2x^2 + \cdots\right) = 0. \end{aligned}$$

If (26) is to be a solution of (24), the coefficient of each power of x in this last result must vanish to make it an identity. Collecting terms involving the same power of x and equating their coefficients to zero will lead to a sequence of equations connecting the coefficients a_n in (26), and equating the coefficient of the lowest power of x to zero will give an equation from which c can be determined.

The lowest power of x in the preceding result is x^{c-2}, so collecting terms involving x^{c-2} and equating the coefficient of x^{c-2} to zero gives

$$[c(c-1) + p_0 c + q_0]a_0 = 0.$$

As Theorem 8.1 requires $a_0 \neq 0$, it follows that c is determined by the equation

$$c(c-1) + p_0 c + q_0 = 0. \tag{30}$$

indicial equation

This equation is called the **indicial equation** associated with differential equation (24), because it determines the permissible values of the index c to be used in the solution given in Theorem 8.1.

The indicial equation of differential equation (24) can be constructed without the need to make the substitution (26), because it is easily seen that

$$p_0 = \lim_{x\to 0}[xP(x)] \quad \text{and} \quad q_0 = \lim_{x\to 0}[x^2 Q(x)]. \tag{31}$$

For the class of equations of type (24) that all have a regular singular point at the origin, the appropriate form of the Frobenius theorem follows from Theorem 8.1 if we set $x_0 = 0$.

It is important to notice that for a general equation (22) in which $a(x) \neq x^2$ the indicial equation does *not* take the form given in (30). When this situation arises the indicial equation must be obtained by substituting (26) into (22) and equating to zero the coefficient of the lowest power of x that occurs in the expansion.

As the indicial equation is a quadratic equation in c, the following relationships between its roots c_1 and c_2 are possible:

(a) Roots c_1 and c_2 are real and distinct and do not differ by an integer
(b) Roots c_1 and c_2 are real and differ by an integer
(c) Roots c_1 and c_2 are real and equal
(d) Roots c_1 and c_2 are complex conjugates

The reason for identifying these different cases is to be found in the following theorem, which is stated without proof in terms of a differential equation with a regular singular point located at the origin (see references [3.3] and [3.5]).

THEOREM 8.2

Forms of Frobenius solution depending on the nature of c_1 and c_2 Let a differential equation of the form

$$x^2 y'' + x[xP(x)]y' + [x^2 Q(x)]y = 0$$

have a regular singular point at $x = 0$. Let $xP(x)$ and $x^2 Q(x)$ each be capable of expansion as convergent power series in an interval $|x| < d$, where $d > 0$ is the

smaller of the two radii of convergence, and suppose that

$$p_0 = \lim_{x\to 0}[xP(x)] \quad \text{and} \quad q_0 = \lim_{x\to 0}[x^2 Q(x)].$$

Then in terms of the exponent c in (26), and the coefficients p_0 and q_0, the indicial equation for the differential equation is

$$c(c-1) + p_0 c + q_0 = 0,$$

with two roots c_1 and c_2 that may be real or complex conjugates.

The two linearly independent solutions of the differential equation that exist depend on the relationship between the roots of the indicial equation, and they take the following forms.

different forms of Frobenius solution and examples

Case (a) Real roots with $c_1 > c_2$ and $c_1 - c_2$ neither zero nor a positive integer

In the intervals $-d < x < 0$ and $0 < x < d$ the differential equation has two linearly independent solutions of the form

$$y_1(x) = |x|^{c_1}\left[1 + \sum_{n=1}^{\infty} a_n x^n\right] \quad \text{and} \quad y_2(x) = |x|^{c_2}\left[1 + \sum_{n=1}^{\infty} b_n x^n\right],$$

where the coefficients a_n are obtained by substituting $c = c_1$ in the recurrence relation connecting coefficients and then setting $a_0 = 1$, and the coefficients b_n are obtained in similar fashion by substituting $c = c_2$ in the recurrence relation, replacing a_n by b_n and setting $b_0 = 1$.

Case (b) Real roots with $c_1 - c_2$ equal to a positive integer

In the intervals $-d < x < 0$ and $0 < x < d$ the differential equation has two linearly independent solutions of the form

$$y_1(x) = |x|^{c_1}\left[1 + \sum_{n=1}^{\infty} a_n x^n\right] \quad \text{and} \quad y_2(x) = Ay_1(x)\ln|x| + |x|^{c_2}\sum_{n=1}^{\infty} \beta_n x^n,$$

where the coefficients a_n are determined as in Case (a), and the coefficients A and β_n are found by substituting $y(x) = y_2(x)$ in the differential equation. Some differential equations for which $c_1 - c_2$ is a positive integer have no logarithmic term in their solution $y_2(x)$, in which case $A = 0$.

Case (c) Real roots with $c_1 = c_2$

In the intervals $-d < x < 0$ and $0 < x < d$ the differential equation has two linearly independent solutions of the form

$$y_1(x) = |x|^{c_1}\left[1 + \sum_{n=1}^{\infty} a_n x^n\right] \quad \text{and} \quad y_2(x) = y_1(x)\ln|x| + |x|^{c_1}\sum_{n=1}^{\infty} \alpha_n x^n,$$

where the coefficients a_n are determined as in Case (a), and the coefficients α_n are found by substituting $y(x) = y_2(x)$ into the differential equation.

Case (d) Complex conjugate roots

If $c_1 = \lambda + i\mu$ and $c_2 = \lambda - i\mu$ with $\mu \neq 0$, then in the intervals $-d < x < 0$ and $0 < x < d$ the two linearly independent solutions of the differential equation are the real and imaginary parts of

$$y(x) = |x|^{\lambda+i\mu}\left[1 + \sum_{n=1}^{\infty} a_n x^n\right],$$

where the coefficients a_n are determined as in Case (a). ■

It is important to recognize that the solutions in cases (a) to (d) of Theorem 8.2 all lie in intervals of the form $0 < x < d$ that do *not* contain the origin. A solution in the interval $-d < x < 0$ can be obtained from the above results by replacing x by $-x$ and, depending on the relationship between the roots c_1 and c_2, seeking a solution in the manner indicated in the illustrative examples that follow.

Case (a) Roots c_1 and c_2 Are Distinct and Do Not Differ by an Integer

EXAMPLE 8.8 Find the solution of

$$2xy'' + (x+1)y' + y = 0$$

in some interval $0 < x < d$.

Solution As the coefficient of y'' vanishes at $x = 0$ the origin must be a singular point of this equation. When the differential equation is written in standard form we find that $P(x) = (x+1)/(2x)$ and $Q(x) = 1/(2x)$, so $p_0 = \lim_{x\to 0} xP(x) = 1/2$ and $q_0 = \lim_{x\to 0} x^2 Q(x) = 0$, showing that the origin is a regular singular point of the differential equation.

From (30) the indicial equation is seen to be

$$c(c-1) + \frac{1}{2}c = 0, \quad \text{or} \quad c\left(c - \frac{1}{2}\right) = 0,$$

showing that the permissible values of c are $c = 0$ and $c = 1/2$. As these values of c are distinct and do not differ by an integer, the solution will be of the type given in Theorem 8.2(a).

Setting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+c}$$

and substituting into the differential equation in the usual way leads to the result

$$2\sum_{n=0}^{\infty}(n+c)(n+c-1)a_n x^{n+c-1} + \sum_{n=0}^{\infty}(n+c)a_n x^{n+c} + \sum_{n=0}^{\infty}(n+c)a_n x^{n+c-1} + \sum_{n=0}^{\infty} a_n x^{n+c} = 0.$$

Shifting the summation index in the first and third summations gives

$$2\sum_{n=-1}^{\infty}(n+c+1)(n+c)a_{n+1}x^{n+c}+\sum_{n=0}^{\infty}(n+c)a_nx^{n+c}+\sum_{n=-1}^{\infty}(n+c+1)a_{n+1}x^{n+c}$$
$$+\sum_{n=0}^{\infty}a_nx^{n+c}=0,$$

and, finally, combining terms we arrive at the result

$$\sum_{n=-1}^{\infty}[2(n+c+1)(n+c)+(n+c+1)]a_{n+1}x^{n+c}+\sum_{n=0}^{\infty}(n+c+1)a_nx^{n+c}=0.$$

Separating out the term corresponding to $n=-1$ allows this to be written

$$[2c(c-1)+c]a_0x^{c-1}+\sum_{n=0}^{\infty}\{[2(n+c+1)(n+c)+(n+c+1)]a_{n+1}$$
$$+(n+c+1)a_n\}x^{n+c}=0.$$

To proceed further we must now equate to zero the coefficient of each power of x. Equating to zero the coefficient of x^{c-1} simply gives the indicial equation, but equating to zero the coefficient of x^{n+c} for $n=0,1,2,\ldots$ gives

$$(n+c+1)(2n+2c+1)a_{n+1}+(n+c+1)a_n=0.$$

As $n+c+1\neq 0$ this recurrence relation can be written

$$a_{n+1}=-\frac{a_n}{2n+2c+1}.$$

Starting with the value $c=0$, we find that

$$a_{n+1}=-\frac{a_n}{2n+1},$$

so

$$a_1=-a_0,\quad a_2=-\frac{a_1}{3}=\frac{a_0}{3},\quad a_3=-\frac{a_2}{5}=-\frac{a_0}{3\cdot 5},\quad a_4=-\frac{a_3}{7}=\frac{a_0}{3\cdot 5\cdot 7},$$
$$a_5=-\frac{a_4}{9}=-\frac{a_0}{3\cdot 5\cdot 7\cdot 9},\quad a_6=-\frac{a_5}{11}=\frac{a_0}{3\cdot 5\cdot 7\cdot 9\cdot 11},\ldots.$$

Examination of a_5 and a_6 shows they can be written

$$a_5=-\frac{2\cdot 4\cdot 6\cdot 8}{9!}a_0=-\frac{2^4\cdot 4!}{(2\cdot 4+1)!}a_0$$

and

$$a_6=\frac{2^5\cdot 5!}{(2\cdot 5+1)!}a_0.$$

These expressions suggest that the coefficient of the general term in the series is

$$a_{n+1}=\frac{(-1)^{n+1}2^nn!}{(2n+1)!}a_0\quad\text{for } n=0,1,2,\ldots,$$

and this is easily verified by mathematical induction. As we are considering the case

in which $c = 0$, it follows from Theorem 8.2(a) that for some $d_1 > 0$ one solution is

$$y(x) = a_0\left[1 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}2^n n!}{(2n+1)!}x^{n+1}\right].$$

As the constant $a_0 \neq 0$ is arbitrary, we set $a_0 = 1$ and take for a fundamental solution of the differential equation

$$y_1(x) = 1 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}2^n n!}{(2n+1)!}x^{n+1} \quad \text{for } 0 < x < d_1.$$

A second fundamental (linearly independent) solution follows by using the other value $c = 1/2$, for which the recurrence relation becomes

$$a_{n+1} = -\frac{a_n}{2n+2}.$$

Using this result and recognizing that the coefficients a_n are not the same as the ones in $y_1(x)$, we find that

$$a_1 = -\frac{a_0}{2}, \quad a_2 = -\frac{a_1}{2\cdot 2} = \frac{a_0}{2^2\cdot 2!}, \quad a_3 = -\frac{a_2}{2\cdot 3} = -\frac{a_0}{2^3\cdot 3!},$$
$$a_4 = -\frac{a_3}{2\cdot 4} = \frac{a_0}{2^4\cdot 4!}, \ldots .$$

This pattern of coefficients suggests that the coefficient of the general term in the series is

$$a_n = \frac{(-1)^n}{2^n n!}a_0,$$

and this also is easily verified by using an inductive argument. Setting the arbitrary constant $a_0 = 1$, it follows from Theorem 8.2(a) that for some $d_2 > 0$ a second fundamental solution is given by

$$y_2(x) = x^{1/2}\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}x^n = x^{1/2}e^{-x/2}, \quad \text{for } 0 < x < d_2.$$

The solutions $y_1(x)$ and $y_2(x)$ form a basis for solutions of the differential equation in an interval of the form $0 < x < d$, where $d = \min\{d_1, d_2\}$. Thus, the general solution is

$$y(x) = C_1 y_1(x) + C_2 y_2(x), \quad \text{for } 0 < x < d,$$

where C_1 and C_2 are arbitrary constants. The value of d is

$$d = \min\{R_1, R_2\},$$

where R_1 and R_2 are the radii of convergence of the series solutions for $y_1(x)$ and $y_2(x)$, respectively. In this case $R_1 = R_2 = \infty$, so the general solution is valid for $x > 0$. ■

Case (b) Roots c_1 and c_2 Are Real and Differ by an Integer

EXAMPLE 8.9 Find the solution of

$$x^2y'' + x(2+x)y' - 2y = 0$$

in some interval $0 < x < d$.

Solution The equation has a singular point at the origin, and writing it in standard form shows that $P(x) = (2 + x)/x$ and $Q(x) = -2/x^2$. Thus,

$$p_0 = \lim_{x\to 0} xP(x) = 2 \quad \text{and} \quad q_0 = \lim_{x\to 0} x^2 Q(x) = -2,$$

so the equation has a regular singular point at the origin. It follows from (30) that the indicial equation is

$$c(c-1) + 2c - 2 = 0, \quad \text{or} \quad c^2 + c - 2 = 0.$$

The permissible values of c are thus $c = -2$ and $c = 1$, and these differ by an integer.

Substituting the series

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+c}$$

into the differential equation gives

$$\sum_{n=0}^{\infty}(n+c)(n+c-1)a_n x^{n+c} + 2\sum_{n=0}^{\infty}(n+c)a_n x^{n+c} + \sum_{n=0}^{\infty}(n+c)a_n x^{n+c+1}$$
$$-2\sum_{n=0}^{\infty} a_n x^{n+c} = 0.$$

Shifting the index in the third summation so it starts from $n = 1$ and separating out the terms multiplied by x^c enables the equation to be written

$$a_0(c^2 + c - 2)x^c + \sum_{n=1}^{\infty}\{[(n+c)(n+c+1) - 2]a_n + (n+c-1)a_{n-1}\}x^{n+c} = 0.$$

Proceeding as usual and equating the coefficient of x^c to zero simply gives the indicial equation, whereas equating the coefficient of x^{n+c} to zero gives the recurrence relation

$$a_n = \frac{(n+c-1)}{[2-(n+c)(n+c+1)]}a_{n-1} \quad \text{for } n = 1, 2, \ldots.$$

Considering the larger root $c = 1$, as required by Theorem 8.2(b), we find that

$$a_n = \frac{n}{[2-(1+n)(2+n)]}a_{n-1} \quad \text{for } n = 1, 2, \ldots.$$

So the first few coefficients are

$$a_1 = -\frac{a_0}{4}, \quad a_2 = \frac{2}{[2-3\cdot 4]}a_1 = \frac{a_0}{4\cdot 5}, \quad a_3 = -\frac{3a_2}{[2-4\cdot 5]} = -\frac{a_0}{4\cdot 5\cdot 6},$$
$$a_4 = \frac{4a_3}{[2-5\cdot 6]} = \frac{a_0}{4\cdot 5\cdot 6\cdot 7}, \ldots.$$

As $c = 1$, setting the arbitrary constant $a_0 = 1$, it follows from Theorem 8.2(b) that for some $d_1 > 0$ a fundamental solution of the differential equation is

$$y_1(x) = x\left(1 - \frac{x}{4} + \frac{x^2}{4\cdot 5} - \frac{x^3}{4\cdot 5\cdot 6} + \frac{x^4}{4\cdot 5\cdot 6\cdot 7} - \cdots\right),$$

or

$$y_1(x) = x - \frac{x^2}{4} + \frac{x^3}{4 \cdot 5} - \frac{x^4}{4 \cdot 5 \cdot 6} + \frac{x^5}{4 \cdot 5 \cdot 6 \cdot 7} - \cdots,$$

with $0 < x < d$.

Theorem 8.2(b) asserts that, corresponding to the smaller root $c = -2$, a second fundamental solution is of the form

$$\begin{aligned} y_2(x) &= Cy_1(x) \ln x + x^{-2} \sum_{n=0}^{\infty} b_n x^n \\ &= Cy_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n-2}. \end{aligned}$$

To determine C and the coefficients b_n, we substitute this solution into the original differential equation, and because the result must be an identity in x, the coefficient of each power of x must vanish.

Differentiation of the foregoing result gives

$$y_2' = Cy_1'(x) \ln x + \frac{Cy_1(x)}{x} + \sum_{n=0}^{\infty} (n-2) b_n x^{n-3}$$

and

$$y_2''(x) = Cy_1''(x) \ln x + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty} (n-2)(n-3) b_n x^{n-4}.$$

Substituting these results into the differential equation and collecting terms leads to the result

$$\begin{aligned} &[x^2 y_1''(x) + x(2+x) y_1'(x) - 2y_1(x)] C \ln x + C[y_1(x) + xy_1(x) + 2xy_1'(x)] \\ &\quad + \sum_{n=0}^{\infty} (n-3)(n-2) b_n x^{n-2} + \sum_{n=0}^{\infty} 2(n-2) b_n x^{n-2} + \sum_{n=0}^{\infty} (n-2) b_n x^{n-1} \\ &\quad - \sum_{n=0}^{\infty} 2 b_n x^{n-2} = 0. \end{aligned}$$

The coefficient of the logarithmic term vanishes, because $y_1(x)$ is a solution of the differential equation, so the equation simplifies to

$$\begin{aligned} &C[y_1(x) + xy_1(x) + 2xy_1'(x)] \\ &\quad + \sum_{n=0}^{\infty} (n-3)(n-2) b_n x^{n-2} + \sum_{n=0}^{\infty} 2(n-2) b_n x^{n-2} + \sum_{n=0}^{\infty} (n-2) b_n x^{n-1} \\ &\quad - \sum_{n=0}^{\infty} 2 b_n x^{n-2} = 0. \end{aligned}$$

The terms corresponding to $n = 0$ cancel, and after shifting the summation index in the third summation, we have

$$C[y_1(x) + xy_1(x) + 2xy_1'(x)] + \sum_{n=1}^{\infty} (n-3)(nb_n + b_{n-1}) x^{n-2} = 0.$$

To find the form of the first group of terms $C[y_1(x) + xy_1(x) + 2xy_1'(x)]$, we must use the series solution for $y_1(x)$. As

$$y_1(x) = x - \frac{x^2}{4} + \frac{x^3}{4 \cdot 5} - \frac{x^4}{4 \cdot 5 \cdot 6} + \cdots,$$

differentiation gives

$$y_1'(x) = 1 - \frac{x}{2} + \frac{3x^2}{20} - \frac{x^3}{30} + \cdots,$$

and so

$$C[y_1(x) + xy_1(x) + 2xy_1'(x)] = 3Cx - \frac{Cx^2}{4} + \frac{Cx^3}{10} - \frac{Cx^4}{40} + \cdots.$$

Using this result in the equation and expanding the first few terms in the summation involving the unknown coefficients b_n shows that

$$\left(3Cx - \frac{Cx^2}{4} + \frac{Cx^3}{10} - \frac{Cx^4}{40} + \cdots\right) - (2b_1 + 2b_0)\frac{1}{x} - (2b_2 + b_1) + (4b_4 + b_3)x^2$$
$$+ (10b_5 + 2b_4)x^3 + (18b_6 + 3b_5)x^4 + (28b_7 + 4b_6)x^5 + (40b_8 + 5b_7)x^6 + \cdots = 0.$$

If we now equate to zero the coefficient of each power of x, we find that

$$b_1 = -b_0, \quad b_2 = -\frac{1}{2}b_1 = \frac{1}{2}b_0, \quad C = 0, \quad b_4 = -\frac{1}{4}b_3,$$
$$b_5 = -\frac{1}{5}b_4 = \frac{1}{4 \cdot 5}b_3, \quad b_6 = -\frac{1}{6}b_5 = -\frac{1}{4 \cdot 5 \cdot 6}b_3, \ldots.$$

The condition $C = 0$ shows that in this case the second linearly independent solution $y_2(x)$ does *not* contain a logarithmic term. The terms b_1 and b_2 are determined as multiples of b_0, and from Theorem 8.2(b) $b_0 \neq 0$, whereas for $n > 3$ all of the terms b_n are seen to be multiples of b_3, which is arbitrary because no equation connects it with b_0. Thus, the solution that has been generated appears to contain *two* arbitrary constants instead of the *one* that would have been expected. Substituting the b_n into the general form of the solution, which with $C = 0$ has reduced to

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-2},$$

gives

$$y_2(x) = b_0\left(\frac{1}{x^2} - \frac{1}{x} + \frac{1}{2}\right) + b_3 x\left(1 - \frac{x}{4} + \frac{x^2}{4 \cdot 5} - \frac{x^3}{4 \cdot 5 \cdot 6} + \frac{x^4}{4 \cdot 5 \cdot 6 \cdot 7} - \cdots\right).$$

The apparent incompatibility caused by the introduction of the two arbitrary constants b_0 and b_3 is now resolved, because the series multiplied by b_3 is simply the first linearly independent solution $y_1(x)$. So, in this case, when seeking the second linearly independent solution we have, in fact, generated a linear combination of the first linearly independent solution $y_1(x)$ and another linearly independent solution given by the expression

$$\frac{1}{x^2} - \frac{1}{x} + \frac{1}{2}.$$

Accordingly, we set $b_3 = 0$ and $b_0 = 1$, and take for the second linearly independent solution

$$y_2(x) = \frac{1}{x^2} - \frac{1}{x} + \frac{1}{2},$$

and since only three terms are involved we see that $y_2(x)$ is defined for $x > 0$.

When closed form solutions such as $y_2(x)$ are obtained, they should always be checked by substitution into the differential equation, and in this case it is easy to check that $y_2(x)$ is, indeed, a solution.

It is a simple matter to show the radius of convergence of the series solution $y_1(x)$ is infinite, so solutions $y_1(x)$ and $y_2(x)$ form a basis for the solution of the differential equation whose general solution is

$$y(x) = C_1 y_1(x) + C_2 y_2(x), \quad \text{for } x > 0,$$

where C_1 and C_2 are arbitrary constants. ■

Case (c) Equal Real Roots $c_1 = c_2$

EXAMPLE 8.10 Find the solution of

$$x^2 y'' + (x^2 - x)y' + y = 0,$$

in some interval $0 < x < d$.

Solution This equation has a singular point at the origin, and when expressed in standard form we see that $P(x) = (x-1)/x$ and $Q(x) = 1/x^2$, so

$$p_0 = \lim_{x \to 0} xP(x) = -1 \quad \text{and} \quad q_0 = \lim_{x \to 0} x^2 Q(x) = 1.$$

Thus, the origin is a regular singular point, and from (30) the indicial equation is seen to be

$$c(c-1) - c + 1 = 0, \quad \text{or} \quad (c-1)^2 = 0,$$

so the roots are $c = 1$ (twice).

Substituting the series

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+c}$$

into the differential equation gives

$$\sum_{n=0}^{\infty} (n+c)(n+c-1)a_n x^{n+c} + \sum_{n=0}^{\infty} (n+c)a_n x^{n+c+1} - \sum_{n=0}^{\infty} (n+c)a_n x^{n+c} + \sum_{n=0}^{\infty} a_n x^{n+c} = 0.$$

Shifting the summation index in the second summation allows it to be written

$$\sum_{n=1}^{\infty} (n+c-1)a_{n-1} x^{n+c},$$

so using this in the preceding equation and separating out the terms corresponding to $n = 0$ we find that

$$a_0[c(c-1) - c + 1]x^c + \sum_{n=1}^{\infty}\{[(n+c)(n+c-2)+1]a_n + (n+c-1)a_{n-1}\}x^{n+c} = 0.$$

As usual, equating the coefficient of x^c to zero gives the indicial equation, and equating the coefficient of x^{n+c} to zero gives the recurrence relation

$$[(n+c)(n+c-2)+1]a_n = -(n+c-1)a_{n-1} \quad \text{for } n = 1, 2, \ldots.$$

Setting $c = 1$ this becomes

$$a_n = -a_{n-1}/n,$$

so

$$a_1 = -a_0, \quad a_2 = -\frac{1}{2}a_0 = \frac{1}{2!}a_0, \quad a_3 = -\frac{1}{3}a_2 = \frac{1}{3!}a_0$$

and, in general,

$$a_n = \frac{(-1)^n}{n!} \quad \text{for } n = 0, 1, 2, \ldots.$$

Setting the arbitrary constant $a_0 = 1$ gives as a fundamental solution of the equation

$$y_1(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{n+1} = xe^{-x}.$$

The series for e^{-x} converges for $x > 0$, so this result is valid for all $x > 0$.

Continuing, we now illustrate two different methods by which a second linearly independent solution may be found.

Method 1. As the form of solution $y_1(x)$ is particularly simple, we will make use of result (35) of Section 6.3 that asserts that if $y_1(x)$ is a solution of the equation

$$y'' + P(x)y' + Q(x)y = 0,$$

an example using the reduction of order method

then a second linearly independent solution is given by the **reduction of order** formula

$$y_2(x) = y_1(x)\int \frac{\exp[-\int P(x)dx]}{[y_1(x)]^2}dx.$$

Substituting for $y_1(x)$ and $P(x)$ gives

$$\int P(x)dx = \int \frac{(x-1)}{x}dx = x - \ln x, \quad \text{so } \exp\left[-\int P(x)dx\right] = xe^{-x}.$$

Thus,

$$y_2(x) = y_1(x)\int \frac{xe^{-x}}{x^2e^{-2x}}dx = y_1(x)\int \frac{e^x}{x}dx.$$

To integrate this result we replace e^x by its series expansion and integrate term by

term to obtain

$$y_2(x) = xe^{-x} \int \left(\frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}{x} \right) dx$$

$$= xe^{-x} \left(\ln x + x + \frac{x^2}{4} + \frac{x^3}{18} + \frac{x^4}{96} + \frac{x^5}{600} + \cdots \right).$$

In order to compare this method with the one that is to follow, we rewrite this result by replacing e^{-x} by the first few terms of its series expansion to give

$$y_2(x) = xe^{-x} \ln x + x \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \right) \left(x + \frac{x^2}{4} + \frac{x^3}{18} + \frac{x^4}{96} + \frac{x^5}{600} + \cdots \right).$$

Multiplying the two series together then shows that for some d_2

$$y_2(x) = xe^{-x} \ln x + \left(x^2 - \frac{3x^3}{4} + \frac{11x^4}{36} - \frac{25x^5}{288} + \cdots \right), \quad \text{for } 0 < x < d_2,$$

where d_2 is the radius of convergence of the bracketed series.

Method 2. Theorem 8.2(c) asserts that the second linearly independent solution has the form

$$y_2(x) = y_1(x) \ln\, x + x^2 \sum_{n=0}^{\infty} b_n x^n = y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+2}.$$

Substituting this result into the differential equation and collecting terms gives

$$[x^2 y_1''(x) + (x^2 - x) y_1'(x) + y_1(x)] \ln\, x + 2xy_1'(x) + xy_1(x) - 2y_1(x)$$

$$+ \sum_{n=0}^{\infty} (n+2)(n+1) b_n x^{n+2} + \sum_{n=0}^{\infty} (n+2) b_n x^{n+3} - \sum_{n=0}^{\infty} (n+2) b_n x^{n+2}$$

$$+ \sum_{n=0}^{\infty} b_n x^{n+2} = 0.$$

Notice that the logarithmic term has vanished because $y_1(x)$ is a solution of the differential equation.

Shifting the summation index in the second summation, we obtain

$$2xy_1'(x) + xy_1(x) - 2y_1(x) + \sum_{n=0}^{\infty} (n+2)(n+1) b_n x^{n+2}$$

$$+ \sum_{n=1}^{\infty} (n+1) b_{n-1} x^{n+2} - \sum_{n=1}^{\infty} (n+2) b_n x^{n+2} + \sum_{n=0}^{\infty} b_n x^{n+2} = 0.$$

Separating out the terms corresponding to $n = 0$ allows this to be written as

$$2xy_1'(x) + xy_1(x) - 2y_1(x) + b_0 x^2 + \sum_{n=1}^{\infty} (n+1)[(n+1) b_n + b_{n-1}] x^{n+2} = 0.$$

The terms involving $y_1(x)$ are now obtained by differentiation of the series

$$y_1(x) = xe^{-x} = x - x^2 + x^3/3 - x^4/6 + x^5/24 - \cdots,$$

leading to

$$2xy_1'(x) + xy_1(x) - 2y_1(x) = -x^2 + x^3 - x^4/2 + x^5/6 - x^6/24 + \cdots.$$

Using this result in the above equation and expanding the terms involving b^n gives

$$(-x^2 + x^3 - x^4/2 + x^5/6 - x^6/24 + \cdots) + b_0x^2 + 2(2b_1 + b_0)x^3$$
$$+ 3(3b_2 + b_1)x^4 + 4(4b_3 + b_2)x^5 + 5(5b_4 + b_3)x^6 + \cdots = 0.$$

Finally, equating the coefficients of powers of x to zero gives

$$b_0 - 1 = 0, \quad 4b_1 + b_0 + 1 = 0, \quad 9b_2 + 3b_1 - 1/2 = 0, \ldots.$$

so that

$$b_0 = 1, \quad b_1 = -3/4, \quad b_2 = 11/36, \quad b_3 = -25/288, \ldots.$$

Substituting these coefficients into the general form of the solution again produces the second solution found by Method 1, though in this case Method 1 was simpler. ■

When the indicial equation has either equal roots or roots differing by an integer, and only the leading terms (the most significant ones) are required in the second linearly independent solution $y_2(x)$, the reduction of order method is often the simplest one to use. This approach is illustrated in the following example, and it is typical of how best to proceed when the integrand in result (35) of Section 6.3 involves a quotient of polynomials.

EXAMPLE 8.11 Find the solution of

$$x^2y'' + (x^3 - x)y' + y = 0$$

in some interval $0 < x < d$.

Solution The equation has a singular point at the origin, and when it is written in standard form, we find that $P(x) = x - 1/x$ and $Q(x) = 1/x^2$. Thus,

$$p_0 = \lim_{x\to 0} xP(x) = -1 \quad \text{and} \quad q_0 = \lim_{x\to 0} x^2Q(x) = 1,$$

so the origin is a regular singular point and the indicial equation is

$$c(1 - c) - c + 1 = 0 \quad \text{or} \quad (c - 1)^2 = 0,$$

with the double root $c = 1$.

Making the substitution $y(x) = \sum_{n=0}^{\infty} a_n x^{n+c}$ in the differential equation gives

$$\sum_{n=0}^{\infty}(n + c)(n + c - 1)a_nx^{n+c} + \sum_{n=0}^{\infty}(n + c)a_nx^{n+c+2} - \sum_{n=0}^{\infty}(n + c)a_nx^{n+c}$$
$$+ \sum_{n=0}^{\infty} a_nx^{n+c} = 0.$$

A shift of the summation index brings this to the form

$$(c^2-2c+1)x^c + c^2x^{c+1} + \sum_{n=2}^{\infty}(n+c)(n+c-1)a_nx^{n+c} + \sum_{n=2}^{\infty}(n+c-2)a_{n-2}x^{n+c}$$
$$-\sum_{n=2}^{\infty}(n+c)a_nx^{n+c} + \sum_{n=2}^{\infty}a_nx^{n+c} = 0,$$

and after combination of the summations this becomes

$$(c^2-2c+1)a_0x^c + c^2a_1x^{c+1} + \sum_{n=2}^{\infty}\{[(n+c)(n+c-2)+1]a_n$$
$$+ (n+c-2)a_{n-2}\}x^{n+c} = 0.$$

Equating the coefficient of x^c to zero gives the indicial equation with the double root $c = 1$, and equating the coefficient of x^{c+1} to zero shows that $a_1 = 0$, because $c = 1$. Equating the coefficient of x^{n+c} to zero leads to the recurrence relation

$$[(n+c)(n+c-2)+1]a_n + (n+c-2)a_{n-2} = 0 \quad \text{for } n \geq 2.$$

Setting $c = 1$ in the recurrence relation, we have

$$a_n = -\frac{(n-1)}{n^2}a_{n-2},$$

but as $a_1 = 0$, it follows immediately that $a_n = 0$ for all odd n. As a result we have

$$a_2 = -\frac{1}{2^2}a_0, \quad a_4 = -\frac{3}{4^2}a_2 = \frac{3}{2^2\cdot 4^2}a_0, \quad a_6 = -\frac{5}{6^2}a_4 = -\frac{1\cdot 3\cdot 5}{2^2\cdot 4^2\cdot 6^2}a_0, \ldots,$$

so a fundamental solution is given by

$$y_1(x) = x\left(1 - \frac{1}{2^2}x^2 + \frac{1\cdot 3}{2^2\cdot 4^2}x^4 - \frac{1\cdot 3\cdot 5}{2^2\cdot 4^2\cdot 6^2}x^6 - \cdots\right),$$

or for $0 < x < d_1$, where d_1 is the radius of convergence of $y_1(x)$, by

$$y_1(x) = x - \frac{1}{2^2}x^3 + \frac{1\cdot 3}{2^2\cdot 4^2}x^5 - \frac{1\cdot 3\cdot 5}{2^2\cdot 4^2\cdot 6^2}x^7 + \cdots.$$

The reduction of order method in (35) of Section 6.3 shows that

$$y_2(x) = y_1(x)\int \frac{\exp\left[-\int P(x)dx\right]}{[y_1(x)]^2}dx,$$

but $\exp[-\int P(x)dx] = \exp(-x^2/2)$, so

$$y_2(x) = y_1(x)\int \frac{\exp(-x^2/2)}{[y_1(x)]^2}dx.$$

To find the leading terms in the expansion for $y_2(x)$ it is now necessary to replace $\exp(-x^2/2)$ and $[y_1(x)]^2$ by the first few terms of their series expansions and then to convert the integrand to a polynomial that can be integrated term by term. We have

$$y_2(x) = y_1(x)\int \frac{x\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \frac{1}{384}x^8 - \cdots\right)}{x^2\left(1 - \frac{1}{4}x^2 + \frac{3}{64}x^4 - \frac{5}{768}x^6 + \frac{35}{49152}x^8 - \cdots\right)^2}dx.$$

If the bracketed term in the denominator is now squared, the integral becomes

$$y_2(x) = y_1(x)\int \frac{(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \frac{1}{384}x^8 - \cdots)}{x(1 - \frac{1}{2}x^2 + \frac{5}{32}x^4 - \frac{7}{192}x^6 + \frac{169}{24576}x^8 - \cdots)}dx.$$

Division of the two polynomials using long division, or writing the numerator as

$$\frac{1}{x}\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \cdots\right)\left(1 - \frac{1}{2}x^2 + \frac{5}{32}x^4 - \cdots\right)^{-1}$$

and multiplying the bracketed terms after using the binomial theorem to expand the second bracket, converts the expression for $y_2(x)$ to

$$y_2(x) = y_1(x)\int \frac{1}{x}\left(1 - \frac{1}{32}x^4 + \frac{5}{8192}x^8 - \cdots\right)dx.$$

Integrating term by term, we find that for some $d_2 > 0$, the first few terms of the series solution $y_2(x)$ are

$$y_2(x) = y_1(x)\left[\ln x - \frac{1}{128}x^4 + \frac{5}{65536}x^8 + \cdots\right],$$

or

$$y_2(x) = y_1(x)\ln x + x\left(1 - \frac{1}{4}x^2 + \frac{3}{64}x^4 - \frac{5}{768}x^6 + \frac{35}{4915}x^8 - \cdots\right)\left(-\frac{1}{128}x^4 + \frac{5}{65536}x^8 + \cdots\right).$$

After multiplication of the two series we obtain

$$y_2(x) = y_1(x)\ln x - \left(\frac{1}{128}x^5 - \frac{59}{65536}x^9 + \cdots\right)$$

in some interval of the form $0 < x < d_2$, where d_2 is the radius of convergence of the bracketed series. The general solution is thus

$$y(x) = C_1y_1(x) + C_2y_2(x), \quad \text{for } 0 < x < d,$$

where C_1 and C_2 are arbitrary constants and $d = \min\{d_1, d_2\}$.

When using this approach it is important to ensure that sufficient terms are retained in the intermediate calculations involving the polynomials for the final result to be accurate to the required power of x. ■

Case (d) Complex Conjugate Roots

EXAMPLE 8.12 Find the solution of the Cauchy–Euler equation

$$x^2y''(x) - xy'(x) + 10y(x) = 0$$

in some interval $0 < x < d$.

Solution This equation has a singular point at the origin, and when expressed in standard form $P(x) = -1/x$ and $Q(x) = 10/x^2$. We have

$$\lim_{x\to 0} xP(x) = -1 \quad \text{and} \quad \lim_{x\to 0} x^2Q(x) = 10,$$

so the origin is a regular singular point. From (30) the indicial equation is seen to be

$$c^2 - 2c + 10 = 0$$

with the complex conjugate roots $c = 1 \pm 3i$. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+c}$$

into the differential equation leads to the result

$$\sum_{n=0}^{\infty}(n+c)(n+c-1)a_n x^{n+c} - \sum_{n=0}^{\infty}(n+c)a_n x^{n+c} + \sum_{n=0}^{\infty} 10a_n x^{n+c} = 0.$$

After terms are collected under a single summation sign, this becomes

$$\sum_{n=0}^{\infty}[(n+c)(n+c-2)+10]a_n x^{n+c} = 0.$$

Equating to zero the coefficient of x^c, corresponding to $n = 0$, gives

$$(c^2 - 2c + 10)a_0 = 0,$$

but by hypothesis $a_0 \neq 0$, so this simply yields the indicial equation. Equating to zero the coefficient of x^{n+c} for $n = 1, 2, \ldots$ gives

$$(n+c)(n+c+10)a_n = 0,$$

but as $c = 1 \pm 3i$, the factor $(n+c)(n+c+10) \neq 0$ for any value of n, so it follows that $a_n = 0$ for $n = 1, 2, \ldots$. Thus, from Theorem 8.2(d), it follows that two linearly independent solutions of the differential equation are obtained by taking the real and imaginary parts of

$$\begin{aligned} y(x) &= a_0 x^{1+3i} = a_0 x \exp\{\ln x^{3i}\} = a_0 x \exp\{3i \ln x\} \\ &= a_0 x\{\cos(3 \ln x) + i \sin(3 \ln x)\}. \end{aligned}$$

Setting the arbitrary constant $a_0 = 1$ and taking the real and imaginary parts of this last result shows that two linearly independent solutions are

$$y_1(x) = x \cos\{3 \ln x\} \quad \text{and} \quad y_2(x) = x \sin\{3 \ln x\},$$

each of which is defined for $x > 0$. These solutions form a basis for the solution of the differential equation whose general solution is

$$y(x) = C_1 x \cos\{3 \ln x\} + C_2 x \sin\{3 \ln x\}, \quad \text{for } x > 0,$$

where C_1 and C_2 are arbitrary constants. ■

More information about singular points and the Frobenius method can be found in references [3.3] to [3.6].

Summary

This section showed how the power series solutions considered previously must be modified if solutions are to be obtained in the form of expansions about regular singular points. The method due to Frobenius for obtaining such solutions was then developed systematically and illustrated by examples, with particular attention being given to the various special cases that arise depending on the relationship that exists between the roots of the indicial equation.